On the Langlands reciprocity and functoriality principles

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Outline of the talk.

Notation.

- $\{G_i\}_{i \in I}$: a collection of k_ω-topological groups, where the index set I is countable.
- For all but finitely many $i \in I$, let O_i be a fixed open subgroup of G_i.
- I_∞: the finite subset of I consisting of all $i \in I$ for which O_i is **not** defined.

References.
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Universal mapping property of free products.

- Let $*_{i \in I} G_i$ denote the free product of the collection $\{G_i\}_{i \in I}$ together with the canonical embeddings

$$\iota_{i_o} : G_{i_o} \hookrightarrow \underset{i \in I}{*} G_i,$$

for each $i_o \in I$.

- **The universal mapping property of free products:** Let H be a topological group s.t. $\forall i_o \in I$, $\exists$ a cont. homomorphism $\phi_{i_o} : G_{i_o} \to H$.
THEN: $\exists!$ cont. homomorphism $\phi : *_{i \in I} G_i \to H$, such that $\phi \circ \iota_{i_o} = \phi_{i_o}$, for every $i_o \in I$.

Definition (Restricted free products of top. groups).

- For every finite subset S of I satisfying $I_\infty \subseteq S$, define the topological group

$$G_S := \mathop{\ast}_{i \notin S} O_i \ast \left(\mathop{\ast}_{i \in S} G_i \right)$$

as the free product of the topological groups O_i, for $i \in I - S$, and G_i, for $i \in S$.

- G_S exists in the category of topological groups.
- For finite subsets S and T of I, such that $I_\infty \subseteq S \subseteq T$, the continuous homomorphism

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$ is defined naturally by the *"universal mapping property of free products"*.

- The *restricted free product of the collection* $\{G_i\}_{i\in I}$ *with respect to the collection* $\{O_i\}_{i\in I-I_\infty}$, which is denoted by $*'_{i\in I}(G_i : O_i)$, is defined by the injective limit

$$\mathop{*}_{i\in I}{}'(G_i : O_i) := \varinjlim_{S} G_S$$

of G_S over all possible such finite $S \subset I$ s.t. $I_\infty \subseteq S$, where the connecting morphism are

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$.
- The topology on $*'_{i\in I}(G_i : O_i)$: defined by declaring $X \subseteq *'_{i\in I}(G_i : O_i)$ to be open if $X \cap G_S$ is open in G_S for every S. So, endowed with this topology, $*'_{i\in I}(G_i : O_i)$ is a topological group. **This is the place where the assumptions that I is countable and $\forall i \in I$, G_i is a k_ω-group are used.**

Universal mapping property of restricted free products.

- Let H be a topological group.
- Assume: $\forall i \in I$, $\exists$ a cont. homomorphism

$$\phi_i : G_i \to H.$$

THEN,

- $\exists!$ cont. homomorphism $\phi_S : G_S \to H$, $\forall$ finite $S \underset{finite}{\subset} I$ s.t. $I_\infty \subseteq S$, and
- $\exists!$ cont. homomorphism $\phi = \varinjlim_S \phi_S : *'_{i\in I}(G_i : O_i) \to H$ satisfying

$$\phi_S = \phi \circ c_S : G_S \xrightarrow{c_S} *'_{i\in I}(G_i : O_i) \xrightarrow{\phi} H,$$

where $c_S : G_S \to *'_{i\in I}(G_i : O_i)$ is the canonical hom., $\forall S$.

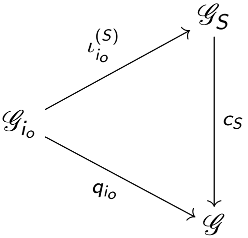
Basic facts.

Let $\mathscr{G} = *'_{i \in I}(G_i : O_i)$. For each $i_o \in I$, set $\mathscr{G}_{i_o} = G_{i_o}$.

- There exists a natural continuous homomorphism

$$q_{i_o} : \mathscr{G}_{i_o} \to \mathscr{G}$$

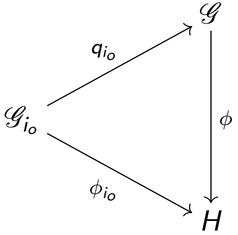
defined explicitly via the commutative triangle



where S is a finite subset of I satisfying $I_\infty \subseteq S$ and $i_o \in S$.

Observation: The homomorphism $q_{i_o} : \mathscr{G}_{i_o} \to \mathscr{G}$ does not depend on the choice of S.

- Moreover, the triangle



  is commutative.
- Let
$$\phi' : \mathscr{G} \to H$$
  be a continuous homomorphism, such that
$$\phi' \circ q_{i_o} = \phi_{i_o},$$
  for every $i_o \in I$. Then,
$$\phi' = \phi.$$

Let G and H be two topological groups, and

$$\xi : G \to H$$

be a continuous homomorphism from G to H. Recall that, a continuous homomorphism

$$\xi' : G \to H$$

from G to H is said to be H-conjugate to $\xi : G \to H$, denoted by $\xi' \sim_H \xi$, if there exists $h \in H$ such that

$$\xi' = \iota_h \circ \xi,$$

where $\iota_h : H \xrightarrow{\sim} H$ is the inner-automorphism of H defined by the h-conjugation as $\iota_h(x) = h^{-1}xh$, for every $x \in H$. Clearly, if $\xi' \sim_H \xi$, then $\ker(\xi') = \ker(\xi)$.

Definition

Let

$$\phi : \mathscr{G} \to H$$

be a continuous homomorphism from $\mathscr{G}$ to H. A continuous homomorphism

$$\phi' : \mathscr{G} \to H$$

is said to be **locally H-conjugate to** $\phi : \mathscr{G} \to H$, if

$$\phi'_{i_o} = \phi' \circ q_{i_o} \sim_H \phi \circ q_{i_o} = \phi_{i_o},$$

for every $i_o \in I$.

- Let
$$\phi, \phi' : \mathscr{G} \to H$$
be two continuous homomorphisms from $\mathscr{G}$ to H. If ϕ and ϕ' are locally H-conjugate, then
$$\ker(\phi) = \ker(\phi').$$

Why restricted free products ?

- Because :

$$*'_{i\in I}(G_i : O_i) \xrightarrow{\mathrm{ab}} (*'_{i\in I}(G_i : O_i))^{\mathrm{ab}} \xrightarrow{\sim} \prod_{i\in I}{}'(G_i^{ab} : O_i^{ab}).$$

  Here, $\prod'_{i\in I}(G_i^{ab} : O_i^{ab})$ is the restricted direct product of the collection $\{G_i^{\mathrm{ab}}\}_{i\in I}$ w.r.t. the collection $\{O_i^{\mathrm{ab}}\}_{i\in I-I_\infty}$.
- Choosing the index set I as the set of places of a global field K and I_∞ as the finite set consisting of all infinite places of K, G_i for $i \in I$, and O_i for $i \in I - I_\infty$ as certain "topological groups defined in terms of the arithmetic attached to the global field K" in such a way that $G_i^{\mathrm{ab}} \simeq K_i^\times$ for all places i of K and $O_i^{\mathrm{ab}} \simeq U_{K_i}$ for all places i of K s.t. $i \notin I_\infty$, this group may be viewed as **a non-commutative generalization of $\mathbb{J}_K$, the idèle group of K.**
- Such a non-abelian generalization of the idèle group $\mathbb{J}_K$ of K is **only possible, if we have a reasonable local non-abelian class field theory over K, in the sense of Hasse, for finite places ν of K**

Notation.

- $K :=$ a number field (or more generally a global field).
- $\mathbf{h}_K = \mathbf{f}_K :=$ the set of all finite places of K.
- $\mathbf{a}_K = \infty_K :=$ the set of all infinite places of K.
- $K_\nu :=$ the ν-adic completion of K at a place ν of K.

References.
[1] J. Arthur, *A note on the automorphic Langlands group*, Canad. Math. Bull., **45**(4), 2002, pp. 466-482.
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Local non-abelian reciprocity map of K_ν $(\nu \in \mathbf{h}_K)$

The groups $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ for $\upsilon \in \mathbf{h}_K$

The aim here is to review very briefly the references [5,6] and [8] .

- For $\nu \in \mathbf{f}_K$, we fix a lifting (=a Lubin-Tate splitting) φ_{K_ν} of the Frobenius automorphism Frob_{K_ν} of K_ν^{nr} to K_ν^{sep}.
- There exists a topological group $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ depending on K_ν, **whose construction uses the theory of APF-extensions and fields of norms of Fontaine-Wintenberger.**
- The topological group $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ comes equipped with a topological group isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Galois}} : \nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \xrightarrow{\sim} G_{K_\upsilon},$$

  we call **the local non-abelian norm residue isomorphism of K_ν in "Galois form"**, because it very much behaves like local abelian norm residue map of K_ν.
- In what follows, we shall consider the "Weil form" of the local non-abelian norm residue isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \xrightarrow{\sim} W_{K_\upsilon},$$

  of K_υ.

Local non-abelian reciprocity map of K_ν ($\nu \in \mathbf{h}_K$)

Ramification filtration on W_{K_ν} in upper numbering

- There exists a subgroup ${}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})^{\underline{o}}}$ of ${}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ such that the "Weil form" of the local non-abelian norm residue arrow $\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \xrightarrow{\sim} W_{K_\upsilon}$ of K_υ induces an isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})^{\underline{o}}} \xrightarrow{\sim} W_{K_\upsilon}^{\underline{o}},$$

  of topological groups (for details look at [6]).

The well-known "local abelian class field theory" and the "local non-abelian class field theory" can be summarized and associated via the following tables :

Local non-abelian reciprocity map of K_ν ($\nu \in \mathbf{h}_K$)

Summary

Non-abelian local C.F.T. (φ_K fixed)	
G_{K_ν}	$\nabla_{K_\nu}^{(\varphi_{K_\nu})}$
W_{K_ν}	${}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}$
$W_{K_\nu}^0$	${}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{0}}}$
$W_{K_\nu}^\delta$, $\delta \in (i-1, i]$	${}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{i}}}$

and **via abelianization**:

Abelian local class field theory	
$G_{K_\nu}^{ab}$	$\widehat{K_\nu^\times}$
$W_{K_\nu}^{ab}$	$K_\nu^\times$
${W_{K_\nu}^{ab}}^0$	U_{K_ν}
${W_{K_\nu}^{ab}}^\delta$, $\delta \in (i-1, i]$	$U_{K_\nu}^i$

The local Langlands group L_{K_ν} of K_ν ($\nu \in \mathbf{h}_K \cup \mathbf{a}_K$)

- The absolute Langlands group L_{K_ν} of K_ν (which exists!) is defined by:
  - $L_{K_\nu} := WA_{K_\nu} := W_{K_\nu} \times \mathrm{SU}(2, \mathbb{R})$, if $\nu \in \mathbf{h}_K$;
  - $L_{K_\nu} := W_{K_\nu}$, if $\nu \in \mathbf{a}_K$,

  where W_{K_ν} denotes the Weil group of K_ν. Recall: $W_{\mathbb{C}} = \mathbb{C}^\times$ and $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ (which is a subgroup of the unit group $\mathbb{H}^\times$ of the division ring of Hamilton quaternions $\mathbb{H}$).
- For $\nu \in \mathbf{h}_K$, fix a Lubin-Tate splitting φ_{K_ν}. The local non-abelian norm residue isomorphism

$$\{\bullet, K_\nu\}^{\mathrm{Weil}}_{\varphi_\nu} : {}_{\mathbb{Z}}\nabla^{(\varphi_{K_\nu})}_{K_\nu} \xrightarrow{\sim} W_{K_\nu}$$

  of K_ν in "Weil form" induces an isomorphism

$$\{\bullet, K_\nu\}^{\mathrm{Langlands}}_{\varphi_\nu} : {}_{\mathbb{Z}}\nabla^{(\varphi_{K_\nu})}_{K_\nu} \times \mathrm{SU}(2, \mathbb{R}) \xrightarrow[\sim]{\{\bullet, K_\nu\}^{\mathrm{Weil}}_{\varphi_{K_\nu}} \times \mathrm{id}_{\mathrm{SU}(2,\mathbb{R})}} L_{K_\nu},$$

  the local non-abelian norm residue isomorphism of K_ν in "Langlands form".

Weil-Arthur idèles of K

Fix $\underline{\varphi} = \{\varphi_{K_\nu}\}_{\nu \in \mathbf{h}_K}$.

- Define an **unconditional** non-commutative topological group $\mathcal{WA}_K^{\underline{\varphi}}$ depending only to the number field K, which we called **the Weil-Arthur idèle group of** K, by the restricted free product

$$\mathcal{WA}_K^{\underline{\varphi}} := \mathop{{\ast}'}_{\nu \in \mathbf{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) : {}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{0}}} \times \mathrm{SU}(2,\mathbb{R}) \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}$$

  of the collection $\{{}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R})\}_{\nu \in \mathbf{h}_K} \cup \{W_{K_\nu}\}_{\nu \in \mathbf{a}_K}$ w.r.t. the collection $\{{}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{0}}} \times \mathrm{SU}(2,\mathbb{R})\}_{\nu \in \mathbf{h}_K}$.
  Here, $r_1 = \#(\mathbf{a}_{K,\mathbb{R}})$ and $2r_2 = \#(\mathbf{a}_{K,\mathbb{C}})$ as usual.
- The topological group $\mathcal{WA}_K^{\underline{\varphi}}$ can be considered as a non-commutative generalization of the idèle group $\mathbb{J}_K$ of K, because $\mathcal{WA}_K^{\underline{\varphi}\, ab} = \mathbb{J}_K$.

Automorphic Langlands group L_K of K

- Let L_K denote the hypothetical automorphic Langlands group L_K of the number field K.
  Assumption: Assume that L_K exists for now.
- It is expected that, an embedding $e_\nu : K^{sep} \hookrightarrow K_\nu^{sep}$ determines a homomorphism (unique up to L_K-conjugacy) $e_\nu^{\mathrm{Langlands}} : L_{K_\nu} \to L_K$.
- Therefore, for $\nu \in \mathbf{h}_K$, there exists a morphism

$$_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) \xrightarrow[\sim]{\{\bullet, K_\nu\}_{\varphi_{K_\nu}}^{\mathrm{Langlands}}} L_{K_\nu} \xrightarrow{e_\nu^{\mathrm{Langlands}}} L_K$$

  (unique up to L_K-conjugacy).

So, by the universal mapping property of restricted free products, we have the following result:

Theorem (The global non-abelian norm residue map of K in "Langlands form")

The collection of arrows $\{e_\nu^{\text{Langlands}} \circ \{\bullet, K_\nu\}_{\varphi_{K_\nu}}^{\text{Langlands}}\}_{\nu \in \mathbf{h}_K}$ *defines a continuous homomorphism*

$$\mathrm{NR}_K^{\varphi^{\text{Langlands}}} : \mathcal{WA}_K^{\varphi} \to L_K,$$

which is unique up to "local L_K-conjugation".

- The arrow $\mathrm{NR}_K^{\varphi^{\text{Langlands}}} : \mathcal{WA}_K^{\varphi} \to L_K$ very much behaves like global abelian norm residue map of K (look at [3]).
- Moreover, this result is compatible with Arthur's construction of L_K (look at [2]).

Conjecture

The homomorphism $\mathrm{NR}_K^{\varphi^{\text{Langlands}}} : \mathcal{WA}_K^{\varphi} \to L_K$ *is open and surjective.*

Notation.

- $\mathcal{N}_K^{\varphi} := \mathrm{Ker}(\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}})$.
- G =a connected reductive group over K.
- $\mathrm{G}_\nu := \mathrm{G} \times_K K_\nu = \mathrm{G}$ as a group over K_ν.
- ${}^L\mathrm{G}(\mathbb{C}) = \widehat{\mathrm{G}}(\mathbb{C}) \rtimes W_K$, the L-group of G over K in "Weil form".
- ${}^L\mathrm{G}_\nu(\mathbb{C}) = \widehat{\mathrm{G}_\nu}(\mathbb{C}) \rtimes W_{K_\nu} = \widehat{\mathrm{G}}(\mathbb{C}) \rtimes W_{K_\nu}$, the L-group of G_ν over K_ν in "Weil form".
- $\eta_\nu : {}^L\mathrm{G}_\nu(\mathbb{C}) \hookrightarrow {}^L\mathrm{G}(\mathbb{C})$ = the natural embedding, given by $e_\nu^{\mathrm{Weil}} : W_{K_\nu} \hookrightarrow W_K$ determined by fixing $e_\nu : K^{sep} \hookrightarrow K_\nu^{sep}$, unique up to W_K-conjugacy.

References.
[1] J. Arthur, *A note on the automorphic Langlands group*, Canad. Math. Bull., **45**(4), 2002, pp. 466-482.
[2] K.I.I., *A note on Arthur's construction of the automorphic Langlands group*, preprint.
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[5] R.P. Langlands, *Beyond endoscopy*, Contributions to automorphic forms, geometry, and number theory, pp. 611–697, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

WA-parameters of G over K

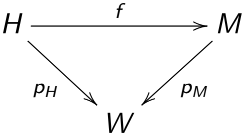
Recall that:

Definition (Morphisms over a group)

Let H and M be groups endowed with group homomorphisms $p_H : H \to W$ and $p_M : M \to W$. A homomorphism

$$f : H \to M$$

is called a **homomorphism over** W, if the following triangle



is commutative.

Definition (Semi-simple elements in ${}^L\mathrm{G}(\mathbb{C})$)

An element $x = (u, \gamma) \in {}^L\mathrm{G}(\mathbb{C})$ is called **semi-simple**, if $r(x)$ is semi-simple in $\mathrm{GL}(n, \mathbb{C})$ for any n-dim. representation r of ${}^L\mathrm{G}(\mathbb{C})$ over $\mathbb{C}$.

Recall that: An n-dimensional representation r of ${}^L\mathrm{G}(\mathbb{C})$ over $\mathbb{C}$, means a continuous homomorphism

$$r : {}^L\mathrm{G}(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

such that

$$r\,|_{\widehat{\mathrm{G}}(\mathbb{C})} : \widehat{\mathrm{G}}(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

is a homomorphism of complex Lie groups.

Definition (Semi-simple homomorphism $H \to {}^L\mathrm{G}(\mathbb{C})$)

Let H be a group. A group homomorphism $f : H \to {}^L\mathrm{G}(\mathbb{C})$ is called **semi-simple**, if $f(h)$ is semi-simple in ${}^L\mathrm{G}(\mathbb{C})$, $\forall h \in H$.

Definition (Equivalent homomorphisms $H \to {}^L\mathrm{G}(\mathbb{C})$)

Let H be a group. Group homomorphisms $f_1, f_2 : H \to {}^L\mathrm{G}(\mathbb{C})$ are called equivalent, denoted $f_1 \sim f_2$, if there exists $g \in \widehat{\mathrm{G}}(\mathbb{C})$ such that :

$$f_2 = \iota_g \circ f_1,$$

where $\iota_g : {}^L\mathrm{G}(\mathbb{C}) \to {}^L\mathrm{G}(\mathbb{C})$ is the inner automorphism of ${}^L\mathrm{G}(\mathbb{C})$ defined by $(g, 1_{W_K}) \in {}^L\mathrm{G}(\mathbb{C})$.

Note that:

- The binary relation "$\sim$" on $\mathrm{Hom}(H, {}^L\mathrm{G}(\mathbb{C}))$ defined via "$\widehat{\mathrm{G}}(\mathbb{C})$-conjugation" is an equiv. relation on $\mathrm{Hom}(H, {}^L\mathrm{G}(\mathbb{C}))$.
- The equivalence class of a homomorphism $f : H \to {}^L\mathrm{G}(\mathbb{C})$ w.r.t "$\sim$" is denoted by $[f]$.
- If $f_1 : H \to {}^L\mathrm{G}(\mathbb{C})$ is a semi-simple homomorphism and $f_1 \sim f_2$, then the homomorphism $f_2 : H \to {}^L\mathrm{G}(\mathbb{C})$ is also semi-simple.
- Let H be a topological group. If $f_1 : H \to {}^L\mathrm{G}(\mathbb{C})$ is a cont. hom. and $f_1 \sim f_2$, then $f_2 : H \to {}^L\mathrm{G}(\mathbb{C})$ is a cont. hom. as well.

Assumption: From now on, assume that G is a **quasisplit** group over K.

Definition (L-homomorphisms)

Let H be a topological group endowed with a continuous homomorphism $p_H : H \to W_K$. A semi-simple and continuous homomorphism

$$f : H \to {}^L\mathrm{G}(\mathbb{C})$$

over W_K is called **an L-homomorphism from H to** ${}^L\mathrm{G}(\mathbb{C})$.

In particular:

Definition (Global L-parameters of G over K)

A semi-simple and continuous group homomorphism

$$\phi : L_K \to {}^L\mathrm{G}(\mathbb{C})$$

over W_K is called **an L-homomorphism from L_K to** ${}^L\mathrm{G}(\mathbb{C})$, and its equivalence class $[\phi]$, consisting of $\widehat{\mathrm{G}}(\mathbb{C})$ -conjugates of ϕ, is called **a global L-parameter of G over K**

Notation

- $\widetilde{\Phi}_K(\mathrm{G})$ = the set of all L-homomorphisms from L_K to ${}^L\mathrm{G}(\mathbb{C})$.
- $\Phi_K(\mathrm{G})$ = the set of all global L-parameters of G over K.

Definition (Local L-parameters of G_ν over K_ν, for $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$)

A semi-simple and continuous group homomorphism

$$\phi_\nu : L_{K_\nu} \to {}^L\mathrm{G}_\nu(\mathbb{C})$$

over W_{K_ν} is called **an L-homomorphism from L_{K_ν} to** ${}^L\mathrm{G}_\nu(\mathbb{C})$, and its equivalence class $[\phi_\nu]$, consisting of $\widehat{\mathrm{G}}(\mathbb{C})$ -conjugates of ϕ_ν, is called **a local L-parameter of G_ν over K_ν, for $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$.**

Notation

- $\widetilde{\Phi}_{K_\nu}(\mathrm{G}_\nu)$ = the set of all L-homomorphisms from L_{K_ν} to ${}^L\mathrm{G}_\nu(\mathbb{C})$, where $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$.
- $\Phi_{K_\nu}(\mathrm{G}_\nu)$ = the set of all local L-parameters of G_ν over K_ν, where $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$.

Remark

If we drop the assumption that G is a quasisplit group over K, then we need an extra condition called **"*G*-relevancy"** in the preceding two definitions.

Definition (*WA*-parameters of G over K)

A semisimple and continuous group homomorphism

$$\phi : \mathcal{WA}_K^{\varphi} \to {}^L\mathrm{G}(\mathbb{C})$$

over W_K is called **an L-homomorphism from $\mathcal{WA}_K^{\varphi}$ to** ${}^L\mathrm{G}(\mathbb{C})$, and its equivalence class $[\phi]$, consisting of $\widehat{\mathrm{G}}(\mathbb{C})$ -conjugates of ϕ, is called **a *WA*-parameter of G over K**.

Notation

- $\widetilde{\Phi}_K^{WA}(\mathrm{G})$ = the set of all L-homomorphisms from $\mathcal{WA}_K^{\varphi}$ to ${}^L\mathrm{G}(\mathbb{C})$.
- $\Phi_K^{WA}(\mathrm{G})$ = the set of all global *WA*-parameters of G over K.

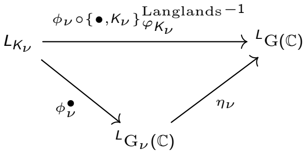
Basic properties:

- Let $\phi : \mathcal{WA}_K^{\varphi} \to {}^L\mathrm{G}(\mathbb{C})$ be an L-homomorphism from $\mathcal{WA}_K^{\varphi}$ to ${}^L\mathrm{G}(\mathbb{C})$. Then, $\forall \nu \in \mathbf{h}_K$, the composition

$$L_{K_\nu} \xrightarrow{\{\bullet, K_\nu\}_{\varphi_{K_\nu}}^{\mathrm{Langlands}^{-1}}} {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) \xrightarrow{\iota_\nu} \mathcal{WA}_K^{\varphi} \xrightarrow{\phi} {}^L\mathrm{G}(\mathbb{C})$$

$$\phi_\nu : {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) \to {}^L\mathrm{G}(\mathbb{C})$$

factors through



where $\phi_\nu^\bullet : L_{K_\nu} \to {}^L\mathrm{G}_\nu(\mathbb{C})$ is an L-homomorphism from L_{K_ν} to ${}^L\mathrm{G}_\nu(\mathbb{C})$.

- So, there exists a mapping
$$\widetilde{\delta}_K^{(G)} : \widetilde{\Phi}_K^{WA}(\mathrm{G}) \to \prod_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \widetilde{\Phi}_{K_\nu}(\mathrm{G}_\nu)$$
defined by
$$\widetilde{\delta}_K^{(G)} : \phi \mapsto \{\phi_\nu^\bullet\}_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K}, \ \forall \phi \in \widetilde{\Phi}_K^{WA}(\mathrm{G}).$$
- The map $\widetilde{\delta}_K^{(G)} : \widetilde{\Phi}_K^{WA}(\mathrm{G}) \to \prod_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \widetilde{\Phi}_{K_\nu}(\mathrm{G}_\nu)$ is an injection.
- If $\phi, \psi \in \widetilde{\Phi}_K^{WA}(\mathrm{G})$ s.t. $\phi \sim \psi$, then $\phi_\nu^\bullet \sim \psi_\nu^\bullet$, $\forall \nu \in \mathbf{h}_K \cup \mathbf{a}_K$.
- Therefore, there exists a mapping
$$\delta_K^{(G)} : \Phi_K^{WA}(\mathrm{G}) \to \prod_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \Phi_{K_\nu}(\mathrm{G}_\nu)$$
defined by
$$\delta_K^{(G)} : [\phi] \mapsto \{[\phi_\nu^\bullet]\}_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K}, \ \forall [\phi] \in \Phi_K^{WA}(\mathrm{G}).$$

- The map $\delta_K^{(G)} : \Phi_K^{WA}(\mathrm{G}) \to \prod_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \Phi_{K_\nu}(\mathrm{G}_\nu)$ is an injection.

Thus, we have:

Theorem (A)

There is an embedding

$$\Phi_K^{WA}(\mathrm{G}) \hookrightarrow \prod_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \Phi_{K_\nu}(\mathrm{G}_\nu)$$

from the set of WA-parameters of G over K into the cartesian product of all local L-parameters of G_ν over K_ν as ν runs over all places in $\mathbf{h}_K \cup \mathbf{a}_K$.

Remark: In fact, it is possible to extend the definition of *WA*-parameters of G over K so that the inclusion stated in Theorem A becomes a bijection.

Global reciprocity principle for G over K

Recall:

$\Pi_{K_\nu}(G_\nu)$ = the set of all equivalence classes of irreducible admissible representations of $G_\nu(K_\nu)$ for $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$.

- **The Local Langlands Reciprocity principle for G_ν over K_ν** for $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$: states that, there exists a partitioning

$$\Pi_{K_\nu}(G_\nu) = \bigsqcup_{[\phi_\nu] \in \Phi_{K_\nu}(G_\nu)} \Pi_{[\phi_\nu]}$$

of $\Pi_{K_\nu}(G_\nu)$ into disjoint finite sets $\Pi_{[\phi_\nu]}$ called ***L*-packets for G_ν over K_ν associated to** $[\phi_\nu] \in \Phi_{K_\nu}(G_\nu)$ in such a way that, the map

$$\mathcal{L}_{K_\nu}^{(G_\nu)} : \Phi_{K_\nu}(G_\nu) \to \Lambda_{K_\nu}(G_\nu)$$

from $\Phi_{K_\nu}(G_\nu)$ to the set $\Lambda_{K_\nu}(G_\nu)$ of all local *L*-packets for G_ν over K_ν, defined by

$$\mathcal{L}_{K_\nu}^{(G_\nu)} : [\phi_\nu] \mapsto \Pi_{[\phi_\nu]}, \quad \forall [\phi_\nu] \in \Phi_{K_\nu}(G_\nu),$$

is the unique bijection satisfying the **"naturality"** properties. Recall that a local L-packet for G_ν over K_ν is a set of all equivalence classes of irreducible admissible representations of $G_\nu(K_\nu)$, which are *L*-**indistinguishable**.

Assumption: From now on, assume that the local Langlands reciprocity principle for G_ν over K_ν holds for every $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$ and for all connected quasisplit reductive groups G over K.

Remark

This is indeed **not** a far-fetched assumption. In fact, as discussed in [3]:

- One of the recent works of L. Fargues and P. Scholze, which is in progress, aims to prove the local Langlands reciprocity conjecture when $\mathrm{char} = 0$.

References for the remark.
[1] A. Genestier and V. Lafforgue, *Chtoucas restraints pour les groupes réductifs et paramétrisation de Langlands locale*, preprint.
[2] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrization de Langlands globale*, J.A.M.S. **31**, 2018, 719–891.
[3] L. A. Lomelí, *Langlands Program and Ramanujan Conjecture: a survey*, A.M.S. Contemporary Mathematics, to appear.
[4] P. Scholze, *The local Langlands correspondence for* GL_n *over p-adic fields*, Invent. Math. **192**, 2013, 663–715.

- **Flath's decomposition theorem:** Let G be a connected reductive group over K. Let π be an irreducible admissible representation of $\mathrm{G}(\mathbb{A}_K)$. Then

$$\pi \simeq \bigotimes_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K}{}' \pi_\nu,$$

where π_ν are irreducible admissible representations of $\mathrm{G}_\nu(K_\nu)$ for all $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$ s.t. all but finitely many π_ν are unramified.

- Now, let $[\phi] \in \Phi_K^{WA}(\mathrm{G})$ be a WA-parameter of G over K. Theorem (A) $\Rightarrow$ $\exists ! [\phi_\nu^\bullet] \in \Phi_{K_\nu}(\mathrm{G}_\nu)$, $\forall \nu \in \mathbf{h}_K \cup \mathbf{a}_K$ s.t.

$$\delta_K^{(G)} : [\phi] \mapsto \{[\phi_\nu^\bullet]\}_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K}.$$

By the local Langlands reciprocity principle for G_ν over K_ν,

$$\mathcal{L}_{K_\nu}^{(G_\nu)} : [\phi_\nu^\bullet] \mapsto \Pi_{[\phi_\nu^\bullet]},$$

where $\Pi_{[\phi_\nu^\bullet]}$ is the L-packet for G_ν over K_ν associated to $[\phi_\nu^\bullet] \in \Phi_{K_\nu}(G_\nu)$, $\forall \nu \in \mathbf{h}_K \cup \mathbf{a}_K$.

- Flath's Theorem $\Rightarrow \bigotimes'_{\nu \in \mathbf{h}_K \cup \mathbf{a}_K} \Pi_{[\phi_\nu^\bullet]}$ is a global admissible L-packet for G over K, which we say from now on, associated to the WA-parameter $[\phi]$, and denote $\Pi_{[\phi]}$.
- $\Lambda_K(G) := \{$global admissible L-packets of G over $K\}$.

We have the following theorem

Theorem (Global Langlands reciprocity principle for G over K.)

Assume that the local Langlands reciprocity principle for G_ν over K_ν holds for all $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$ and for all connected quasisplit reductive groups G over K. Then:

- *There exists a bijection*

$$\mathcal{L}_K(G) : \Phi_K^{WA}(\mathrm{G}) \leftrightarrow \Lambda_K(G)$$

satisfying the "naturality" properties.

- *Assuming the existence of L_K and the Conjecture introduced above,*

$$\Pi_{[\phi]} : \textit{automorphic } L\textit{-packet} \Leftrightarrow \mathcal{N}_K^{\varphi} \subseteq \mathrm{Ker}(\phi).$$

Global functoriality principle over K

Let H be a connected and quasisplit reductive group over K.

Theorem (Global Langlands functoriality principle over K)

Assume that the local Langlands reciprocity principle for G_ν over K_ν and for H_ν over K_ν hold for each $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$ and for all connected quasisplit reductive groups G and H over K. Let

$$\rho : {}^L\mathrm{G}(\mathbb{C}) \to {}^L\mathrm{H}(\mathbb{C})$$

be an L-homomorphism. Then:

- *There exists an associated "lifting"*

$$\rho^* : \Lambda_K(G) \to \Lambda_K(H)$$

from $\Lambda_K(G)$ to $\Lambda_K(H)$ satisfying the "naturality" properties.

Theorem (Continuation)

Moreover, let
$\Lambda_K^{aut}(G) := \{$*global automorphic L-packets of* G *over* $K\}$.

- *Assuming the existence of* L_K *and the Conjecture introduced above yields the global functoriality principle. That is,*

$$\Pi \in \Lambda_K^{aut}(G) \Rightarrow \rho^*(\Pi) \in \Lambda_K^{aut}(H).$$

A dream

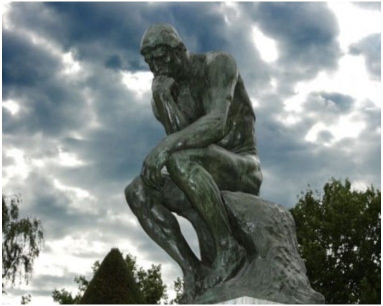
In this talk we tried to say something new about the global reciprocity principle for all connected quasisplit reductive groups G over K and the global functoriality principle over K using only the topological group $\mathcal{WA}_K^{\varphi}$ and under the assumption of the local reciprocity principle for G_ν over K_ν, for all $\nu \in \mathbf{h}_K \cup \mathbf{a}_K$ and for all connected quasisplit reductive groups G over K together with our posed "conjecture". Moreover:

- Seems possible to give an unconditional constructon of the locally compact group L_K attached to the number field K in terms of $\mathcal{WA}_K^{\varphi}$ by introducing the sub-collection of all "automorphic kernels" in the collection of all closed normal subgroups of $\mathcal{WA}_K^{\varphi}$, then proving that the sub-collection of all "automorphic kernels" is a filter for the poset of closed normal subgroups of $\mathcal{WA}_K^{\varphi}$.
- Automorphic existence theorem for Langlands reciprocity principle for G over K

Seems possible to extend the theory and ideas outlined in this talk to the "higher-dimensional" (including the geometric Langlands) to the "dg" setting, and may be to the topological setting. Namely, first we have to develop:

- The local non-abelian Kato-Parshin class field theory (ongoing joint work with E.Serbest);
- "Restricted free product version" of Parshin-Beilinson idèles;
- Extension of "Kapranov reciprocity principle for $\mathrm{GL}(n)$" to arbitrary reductive group G.

—o—



Thank you for your attention!