Modularity, Rational Points and Diophantine Equations

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- $E[p] \subset E(\bar{\mathbb{Q}})$ (the torsion points are algebraic).
- $G_{\mathbb{Q}}$ acts on $E[p]$. We obtain a representation

$$\rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p)$$

Strategy of The Proof of FLT

Theorem (Wiles, Taylor-Wiles)

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  There is a special modular function $g(z) = \sum_{n=1}^{\infty} a_n q^n$ such that $a_q \equiv a_q(E) \pmod{p}$.

Frey Curve and Irreducibility

Let (a, b, c) be a non-trivial primitive solution to the Fermat equation with exponent $p \geq 5$. Let

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A key ingredient: for big enough p and for any E:

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is **irreducible** i.e. NOT upper triangular.

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Given p there exists an algebraic curve, $\mathrm{X}_0(\mathrm{p})$ s.t. :

$$\{\text{Points on } \mathrm{X}_0(\mathrm{p})\}$$

$$\Updownarrow$$

$$\{\rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p) \text{ such that } \rho_{E,p} \sim \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\}$$

Understanding $X_0(N)(\mathbb{Q})$

Example (Equation of $X_0(43)$)

$$f(x,y) = x^4 + 2x^3y + 2x^2y^2 + xy^3 - 2x^2y - 2xy^2 - y^3 + 4x^2 +$$

$$+4xy + 2y^2 - 3y + 4 = 0$$

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If $N > 163$ and prime then $\mathrm{X}_0(\mathrm{N})(\mathbb{Q})$ consists of only cusps.

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Later this has been generalized to composite levels and the situation for small levels is also understood by Kenku, Momose.

What about other Diophantine Equations?

- Can we understand rational solutions $x^p + y^q + z^r = 0$ or $Ax^p + By^p + Cz^p = 0$ or $Ax^p + By^q + Cz^r = 0$ using similar techniques?

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- Modularity of elliptic curves defined over larger fields than $\mathbb{Q}$ OR
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$X_1(N)(K)$ is well understood:

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- ◇ $X_0(N)(K) \Leftrightarrow$ reducible Galois representations OR

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- By Mazur's work: $X_1(N)(\mathbb{Q}) = \{\text{cusps}\}$ if its genus> 1
- Merel: Say $|K : \mathbb{Q}| \leq d$, then there exists B_d such that $X_1(N)(K) = \{\text{cusps}\}$ if $N > B_d$.
- More precise results by Kamienny, Parent, Derickx, Stein, Stoll...

Unfortunately not much is known for $X_0(N)(K)$ except the following:

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- **Bruin, Najman:** parametrized all quadratic points on $X_0(N)$ such that $J_0(N)$ has MW rank 0 and $X_0(N)$ is hyperelliptic:

$$\{22, 23, 26, 28, 29, 30, 31, 33, 35, 39, 40, 41, 46, 47, 48, 50, 59, 71\}$$

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Theorem (O., Siksek)

Found and parametrized all quadratic points on $\mathrm{X}_0(\mathrm{N})$ such that

- $J_0(N)$ *has MW rank* 0 *and*
- $3 \leq g(\mathrm{X}_0(\mathrm{N})) \leq 5$.

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Recently:

Theorem (Box)

All quadratic points on all $\mathrm{X}_0(\mathrm{N})$ *of genus* $2, 3, 4, 5$ *whose Mordell–Weil group has positive rank have been determined. The values of* N *are* $37, 43, 53, 61, 57, 65, 67$ *and* 73.

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- If the mod p representation is reducible, then the Frey curve gives rise to a point in $\mathrm{X}_0(2p)(K)$ or $\mathrm{X}_0(3p)(K)$

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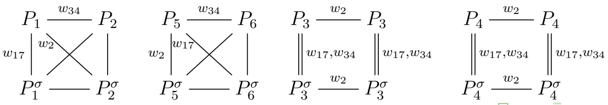
Genus: 3

Model: $x^3z - x^2y^2 - 3x^2z^2 + 2xz^3 + 3xy^2z - 3xyz^2 + 4xz^3 - y^4 + 4y^3z - 6x^2z^2 + 4yz^3 - 2z^4$

$J_0(34)(\mathbb{Q}) = C \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$

Name	θ^2	Coordinates	j-invariant	CM by	$\mathbb{Q}$-curve
P_1	-1	$(\theta+1,0,1)$	287496	-16	YES
P_2	-1	$(\frac{\theta+1}{2},\frac{\theta+1}{2},1)$	1728	-4	YES
P_3	-1	$(\theta,-\theta,1)$	1728	-4	YES
P_4	-2	$(\frac{\theta}{2},-\frac{\theta}{2},1)$	8000	-8	YES
P_5	-15	$(\frac{\theta+11}{8},\frac{1}{2},1)$	$\frac{20410\theta+11779}{8}$	NO	YES
P_6	-15	$(\frac{\theta+23}{16},\frac{\theta+7}{16},1)$	$\frac{-531847853404790\theta-7319387769191}{34359738368}$	NO	YES



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Theorem (O., Siksek)

Found and parametrized all quadratic points on $\mathrm{X}_0(\mathrm{N})$ *such that* $J_0(N)$ *has MW rank* 0*,* $\mathrm{X}_0(\mathrm{N})$ *nonhyperelliptic and* $3 \leq g \leq 5$*.*

$$\{34, 45, 64, 38, 44, 54, 81, 42, 51, 52, 55, 56, 63, 72, 75\}$$

- Say $X/\mathbb{Q}$ is nonhyperelliptic with $g \geq 3$, $J_X(\mathbb{Q})$ is finite and there exists a $P_0 \in X(\mathbb{Q})$.
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$$\iota : X^{(2)}(\mathbb{Q}) \hookrightarrow J_X(\mathbb{Q}), \ P \mapsto [D_P - 2P_0]$$

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5. only then use Riemann Roch.

Final Word of 1st Motivation

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$\mathrm{X}_0(\mathrm{N})(\mathbb{Q}(\sqrt{d}))$ *consists of only cusps if*

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Open Question:

Is there a bound B such that for all $|d| > B, \mathrm{X}_0(\mathrm{N})$ doesn't have any quadratic points for any N? (Say genus of $\mathrm{X}_0(\mathrm{N}) > 2$)

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Application to Diophantine Eqns.

Recall: If (a, b, c) is a solution to FLT_p then:

$y^2 = x(x - a^p)(x + b^p)$ would have very unusual properties.

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There are no three positive integers A, B, and C which satisfy the equation $A^4 + B^2 = C^p$ for any value of p greater than 211.

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How to parametrize $\mathbb{Q}$-curves?

Twists of $X_0(N)$

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Definition

Given a curve X over $\mathbb{Q}$ its *twist* is another curve over $\mathbb{Q}$ that is isomorphic to X over $\bar{\mathbb{Q}}$.

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Remark

Geometrically a curve and its twist are the same.
but arithmetically not... action of Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ differs.

Twists$(X/K) \Leftrightarrow H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{Aut}(X))$

Twist of $X_0(N)$

Recall that $P \in X_0(N) \Leftrightarrow (E, \phi : E \to E')$, $\ker \phi \cong \mathbb{Z}/N\mathbb{Z}$

Definition

Involution w_N on $X_0(N)$: $(E, \phi : E \to E') \mapsto (E', \hat{\phi} : E' \to E)$

Given $K := \mathbb{Q}(\sqrt{d})$, $\mathrm{Gal}(K/\mathbb{Q}) = \langle \tau \rangle$ and $\zeta_d : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(X_0(N))$ be the cocycle that sends τ to w_N.

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Definition

The curve $X_0^d(N)$ is the twist of $X_0(N)$ by ζ_d. In particular rational points of $X_0^d(N)$ are K-rational points of $X_0(N)$ fixed by $\tau \circ w_N$.

- $X_0(N)$ and $X_0^d(N)$ are isomorphic over K.

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Involution w_N on $X_0(N)$: $(E, \phi : E \to E') \mapsto (E', \hat{\phi} : E' \to E)$

Given $K := \mathbb{Q}(\sqrt{d})$, $\mathrm{Gal}(K/\mathbb{Q}) = \langle \tau \rangle$ and
$\zeta_d : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(X_0(N))$ be the cocycle that sends τ to w_N.
Redefine the action of τ on $X_0(N)$ as: $P^\tau := w_N \circ \tau(P)$.

Definition

The curve $X_0^d(N)$ is the twist of $X_0(N)$ by ζ_d. In particular rational points of $X_0^d(N)$ are K-rational points of $X_0(N)$ fixed by $\tau \circ w_N$.

- $X_0(N)$ and $X_0^d(N)$ are isomorphic over K.
- $X_0^d(N)$ is also a moduli space!

$X_0^d(N)(\mathbb{Q}) \Leftrightarrow$ Quadratic $\mathbb{Q}$-curves of degree N defined over K

Field of Definition

Let E be a quadratic $\mathbb{Q}$-curve of degree N defined over $\mathbb{Q}(\sqrt{d})$ then:

E corresponds to a $P \in \mathrm{X}_0^{\mathrm{d}}(\mathrm{N})(\mathbb{Q})$.

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Quick answer: *Not for all d and N*.

Remark: $\mathrm{X}_0^{\mathrm{d}}(\mathrm{N})(\mathbb{Q}) \subset \mathrm{X}_0(\mathrm{N})(\mathbb{Q}(\sqrt{d}))$.
$\mathbb{Q}$-pnts of $\mathrm{X}_0^{\mathrm{d}}(\mathrm{N})$ are K-rational pnts of $\mathrm{X}_0(\mathrm{N})$ fixed by $\tau \circ w_N$.

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If a curve C has real and $\mathbb{Q}_p$-points for every prime p but no $\mathbb{Q}$-points then we say that C violates **the Hasse Principle.**

A conic never violates the Hasse Principle but for higher genus curves there are many examples of the violation.

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Example (Quadratic twists)

Given a prime number $N \equiv 1 \bmod 4$ and a positive integer Z, let A be the set of positive squarefree integers $d \leq Z$ such that the quadratic twist by $\mathrm{K} = \mathbb{Q}(\sqrt{d})$ and w_N violates the Hasse principle. If $N > 131$, the size of A is asymptotically $C_N \frac{Z}{\log^{1-\alpha_N} Z}$.

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- If P in $X(\mathbb{Q})$ then $\mathrm{red}([P]-D)$ is in $\mathrm{inj}(X(\mathbb{F}_p))$ for any p in S.
- If images of red and inj do not intersect then $X(\mathbb{Q}) = \emptyset$.

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$$\mathrm{X_0^d(N)}(\mathbb{Q}) = \{P \in \mathrm{X_0(N)}(\mathbb{Q}(\sqrt{d}))|\tau(P) = -P + S\}$$

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$\mathrm{X_0^d(N)}$ violates the Hasse principle if and only if there exists a local-global trace obstruction for S.

Trace Question

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- *Let $\mathrm{E}/\mathbb{Q}$ be an elliptic curve with non-trivial rational 2-torsion. Under some technical assumptions, the local-global trace principle holds for $\mathrm{E}(\mathbb{Q})_2$ if and only if $\mathrm{E}(\mathrm{K})_2 \neq \mathrm{E}(\mathbb{Q})_2$ or $2\mathrm{E}^d(\mathbb{Q}) \neq \mathrm{E}^d(\mathbb{Q}) \cap 2\mathrm{E}^d(\mathrm{K})$.*

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- *If the local-global trace principle holds for $\mathrm{E}(\mathbb{Q})_2$, then a local trace P is a global trace if and only if certain points in the quadratic twist of E are 2-divisible.*

Trace Question for points that are not 2 torsion

Theorem (Çiperiani, O.)

Let $P \in \mathrm{E}(\mathbb{Q})$ be a local trace at all primes. If the local to global trace criterion holds for $\mathrm{E}(\mathbb{Q})_2$ then P is a global trace if and only if $i(P) \in 2\mathrm{E}^d(\mathrm{K}) + \mathrm{E}^d(\mathbb{Q})$ where $i : \mathrm{E} \to \mathrm{E}^d$ is the isomorphism over K.

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- Check if there is trace obstruction for 2-torsion points.
- If not, then there is no trace obstruction for a non-torsion point P if and only if $i(P)$ is 2-divisible over K nontrivially.

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Epilogue From Barry Mazur:

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Epilogue From Barry Mazur: *'Number theory produces, without effort, innumerable problems which have a sweet, innocent air about them, tempting flowers; and yet ... number theory swarms with bugs, waiting to bite the tempted flower-lovers who, once bitten, are inspired to excess of effort!. '*