Higher moments of distribution of zeta zeros

Farzad Aryan

University of Gottingen

Introduction

- For $s \in \mathbb{C}$, we let

$$B(s) = \sum_{n=1}^{M} \frac{b(n)}{n^s},$$

to be a Dirichlet polynomial.

Introduction

- For $s \in \mathbb{C}$, we let
$$B(s) = \sum_{n=1}^{M} \frac{b(n)}{n^s},$$
to be a Dirichlet polynomial.

- Setting $s = \frac{1}{2} + it$, we can look at $B(\frac{1}{2} + it)$ as a one variable function, with real and imaginary parts.

Introduction

- For $s \in \mathbb{C}$, we let
$$B(s) = \sum_{n=1}^{M} \frac{b(n)}{n^s},$$
to be a Dirichlet polynomial.
- Setting $s = \frac{1}{2} + it$, we can look at $B(\frac{1}{2} + it)$ as a one variable function, with real and imaginary parts.
- Dirichlet polynomials are important subjects of studies in number theory.

- The reason is that they often can mimic the behaviour of L-functions.

- The reason is that they often can mimic the behaviour of L-functions. Many L-functions have series expansion that converge in a section of the complex plane.

- The reason is that they often can mimic the behaviour of L-functions. Many L-functions have series expansion that converge in a section of the complex plane. A finite truncation of the series expansion gives us a Dirichlet polynomial that we can study.

Example

A classical approximation of the Riemann zeta functions gives

$$\zeta(\tfrac{1}{2} + it) = \sum_{n<T} \frac{1}{n^{\frac{1}{2}+it}} - \frac{T^{\frac{1}{2}-it}}{\frac{1}{2} - it} + O(T^{-\frac{1}{2}}), \tag{1}$$

which is valid for $t \leq T$.

- The reason is that they often can mimic the behaviour of L-functions. Many L-functions have series expansion that converge in a section of the complex plane. A finite truncation of the series expansion gives us a Dirichlet polynomial that we can study.

Example

A classical approximation of the Riemann zeta functions gives

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<T} \frac{1}{n^{\frac{1}{2}+it}} - \frac{T^{\frac{1}{2}-it}}{\frac{1}{2}-it} + O(T^{-\frac{1}{2}}), \qquad (1)$$

which is valid for $t \leq T$. Now if $t \sim T$, we get

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<T} \frac{1}{n^{\frac{1}{2}+it}} + O(T^{-\frac{1}{2}}).$$

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.
- I will give more examples of this as we proceed.

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.
- I will give more examples of this as we proceed.
- We write $B(\frac{1}{2}+it)=\Re(B(\frac{1}{2}+it))+i\Im(B(\frac{1}{2}+it))$, and we look at the real and the imaginary part as real functions separately.
- If we like to understand a behaviour of a function, we can start by asking some statistical type questions.

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.
- I will give more examples of this as we proceed.
- We write $B(\frac{1}{2}+it) = \Re(B(\frac{1}{2}+it)) + i\Im(B(\frac{1}{2}+it))$, and we look at the real and the imaginary part as real functions separately.
- If we like to understand a behaviour of a function, we can start by asking some statistical type questions. For example, what is the mean of the function in the interval $[T, 2T]$?

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.
- I will give more examples of this as we proceed.
- We write $B(\frac{1}{2}+it) = \Re(B(\frac{1}{2}+it)) + i\Im(B(\frac{1}{2}+it))$, and we look at the real and the imaginary part as real functions separately.
- If we like to understand a behaviour of a function, we can start by asking some statistical type questions. For example, what is the mean of the function in the interval $[T, 2T]$? Or what is the variance of the function with respect to Lebesgue measure or measures that are relevent to the underlying questions.

- In many cases our problems in the theory of L-functions can brought down to understanding $B(\frac{1}{2}+it)$ with a certain coefficients and t in some range.
- I will give more examples of this as we proceed.
- We write $B(\frac{1}{2}+it) = \Re(B(\frac{1}{2}+it)) + i\Im(B(\frac{1}{2}+it))$, and we look at the real and the imaginary part as real functions separately.
- If we like to understand a behaviour of a function, we can start by asking some statistical type questions. For example, what is the mean of the function in the interval $[T, 2T]$? Or what is the variance of the function with respect to Lebesgue measure or measures that are relevent to the underlying questions. In general we are asking how the valuse of our functions being distribted?

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.
- Mean of $B(\frac{1}{2}+it)$ is calculable, but it does not provide us with much.

$$\frac{1}{T}\int_T^{2T} B(\tfrac{1}{2}+it)dt = \frac{1}{T}\int_T^{2T}\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}} = b(1) + O(\tfrac{\sqrt{M}}{T}),$$

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.
- Mean of $B(\frac{1}{2}+it)$ is calculable, but it does not provide us with much.

$$\frac{1}{T}\int_T^{2T} B(\tfrac{1}{2}+it)dt = \frac{1}{T}\int_T^{2T}\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}} = b(1)+O(\tfrac{\sqrt{M}}{T}),$$

provide that $b(n)$ is not too large (like $b(n) \ll n^{\epsilon}$).

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.
- Mean of $B(\frac{1}{2}+it)$ is calculable, but it does not provide us with much.

$$\frac{1}{T}\int_{T}^{2T} B(\tfrac{1}{2}+it)dt = \frac{1}{T}\int_{T}^{2T}\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}} = b(1) + O(\tfrac{\sqrt{M}}{T}),$$

  provide that $b(n)$ is not too large (like $b(n) \ll n^{\epsilon}$).
- Knowing the variance can be useful in answering some questions, but calculating the variance starts to get tricky and some limitations start to get into the picture.

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.
- Mean of $B(\frac{1}{2}+it)$ is calculable, but it does not provide us with much.

$$\frac{1}{T}\int_T^{2T} B(\tfrac{1}{2}+it)dt = \frac{1}{T}\int_T^{2T}\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}} = b(1)+O(\tfrac{\sqrt{M}}{T}),$$

provide that $b(n)$ is not too large (like $b(n) \ll n^{\epsilon}$).

- Knowing the variance can be useful in answering some questions, but calculating the variance starts to get tricky and some limitations start to get into the picture.
- First of all, the variance is calculable if the length of the Dirichlet polynomials is smaller than the length of the integral.

- Now let me tell you how much we can answer these questions in case of Dirichlet polynomials.
- Mean of $B(\frac{1}{2}+it)$ is calculable, but it does not provide us with much.

$$\frac{1}{T}\int_T^{2T} B(\tfrac{1}{2}+it)dt = \frac{1}{T}\int_T^{2T}\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}} = b(1)+O(\tfrac{\sqrt{M}}{T}),$$

provide that $b(n)$ is not too large (like $b(n) \ll n^{\epsilon}$).

- Knowing the variance can be useful in answering some questions, but calculating the variance starts to get tricky and some limitations start to get into the picture.
- First of all, the variance is calculable if the length of the Dirichlet polynomials is smaller than the length of the integral. If we have $M = o(T)$, then by Montgomery-Vaughan theorem

$$\frac{1}{T}\int_T^{2T}\left|\sum_{n<M}\frac{b(n)}{n^{\frac{1}{2}+it}}\right|^2 \sim \sum_{n<M}\frac{|b(n)|^2}{n}.$$

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

- This is often turnes to an even harder problem than the original.

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

- This is often turnes to an even harder problem than the original.
- For higher moments we can apply the same logic.

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

- This is often turnes to an even harder problem than the original.
- For higher moments we can apply the same logic. For example, for the 4-th moment we need that the length of B must be smaller than $\sqrt{T}$,

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

- This is often turnes to an even harder problem than the original.
- For higher moments we can apply the same logic. For example, for the 4-th moment we need that the length of B must be smaller than $\sqrt{T}$, and for the k-th moment we need $M < T^{2/k}$.

Large Dirichlet polynomilas

- If $M > T$, then to evaluate the variance we need to know about the shifted convolution sums of $b(n)$:

$$\sum_{n<x} b(n)b(n+h).$$

- This is often turnes to an even harder problem than the original.
- For higher moments we can apply the same logic. For example, for the 4-th moment we need that the length of B must be smaller than $\sqrt{T}$, and for the k-th moment we need $M < T^{2/k}$. These are all with respect to the Lebesgue measure.

Other measures

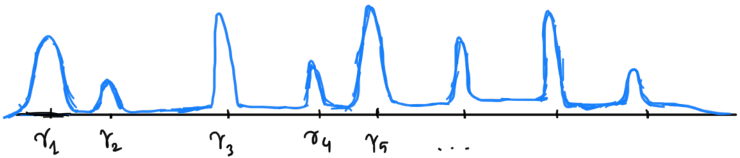
- To understand our function in a better way we can change the measure.

Other measures

- To understand our function in a better way we can change the measure. For example if we are interested in distribution of values of $B(\frac{1}{2}+it)$ when t equals to imaginary part of the zeros of the Riemann zeta functions.

Other measures

- To understand our function in a better way we can change the measure. For example if we are interested in distribution of values of $B(\frac{1}{2} + it)$ when t equals to imaginary part of the zeros of the Riemann zeta functions. Then instead of using Lebesgue measure that gives equal share zeros and non zeros, we can use measures that highlight zeros.
- Imagin that we have a function that look likes



- What we would like to know is the expectation of B^k with respect to this measure:

$$\int |B(\tfrac{1}{2}+it)|^k d\mu.$$

- What we would like to know is the expectation of B^k with respect to this measure:

$$\int |B(\tfrac{1}{2} + it)|^k d\mu.$$

- Therefore first we would like to know how to build such measures.

- What we would like to know is the expectation of B^k with respect to this measure:

$$\int |B(\tfrac{1}{2} + it)|^k d\mu.$$

- Therefore first we would like to know how to build such measures. Also what are the possibilities and limitations.

- What we would like to know is the expectation of B^k with respect to this measure:

$$\int |B(\tfrac{1}{2} + it)|^k d\mu.$$

- Therefore first we would like to know how to build such measures. Also what are the possibilities and limitations.
- A commonly used method is to build the measure by using Dirichlet polynomials.

- What we would like to know is the expectation of B^k with respect to this measure:

$$\int |B(\tfrac{1}{2}+it)|^k d\mu.$$

- Therefore first we would like to know how to build such measures. Also what are the possibilities and limitations.
- A commonly used method is to build the measure by using Dirichlet polynomials.

Definition

Let $A(s) = \sum a(n)n^{-s}$, be a Dirichlet polynomial. Define

$$\mu_A((a,b]) := \frac{\int_a^b |A(\frac{1}{2}+it)|^2 dt}{\int_T^{2T} |A(\frac{1}{2}+it)|^2 dt} \tag{2}$$

- If we set $A(s) = 1$, then μ gives the probability version of the Lebesgue measure.

- If we set $A(s) = 1$, then μ gives the probability version of the Lebesgue measure.

Example

One function that is known to be centered around the zeros of the Riemann zeta function is with comes from

$$A(s) = \sum_{n<T} \frac{\lambda(n)}{n^s},$$

where λ is the Liouville function, defined with

$$\lambda(n) = (-1)^{\#\text{ of prime divisor of } n}.$$

- If we set $A(s) = 1$, then μ gives the probability version of the Lebesgue measure.

Example

One function that is known to be centered around the zeros of the Riemann zeta function is with comes from

$$A(s) = \sum_{n<T} \frac{\lambda(n)}{n^s},$$

where λ is the Liouville function, defined with

$$\lambda(n) = (-1)^{\#\text{ of prime divisor of } n}.$$

In this case $|A(\frac{1}{2} + it)|^2$ would look like the picture I draw before.

The Mean with respect to μ

- Question is how much information we can get from these measures and what are the limitations?

The Mean with respect to μ

- Question is how much information we can get from these measures and what are the limitations?
- Back to the definition of $A(s) = \sum a(n)n^{-s}$, and let B is our original Dirichlet polynomial of arbitrary length.

The Mean with respect to μ

- Question is how much information we can get from these measures and what are the limitations?
- Back to the definition of $A(s) = \sum a(n)n^{-s}$, and let B is our original Dirichlet polynomial of arbitrary length.
- We first look at the first moment (the mean) i.e.

$$\frac{\int_T^{2T} B(\frac{1}{2}+it)|A(\frac{1}{2}+it)|^2 dt}{\int_T^{2T} |A(\frac{1}{2}+it)|^2 dt}$$

The Mean with respect to μ

- Question is how much information we can get from these measures and what are the limitations?
- Back to the definition of $A(s) = \sum a(n)n^{-s}$, and let B is our original Dirichlet polynomial of arbitrary length.
- We first look at the first moment (the mean) i.e.

$$\frac{\int_T^{2T} B(\frac{1}{2}+it)|A(\frac{1}{2}+it)|^2 dt}{\int_T^{2T} |A(\frac{1}{2}+it)|^2 dt} = \int B(\tfrac{1}{2}+it) d\mu_A$$

The Mean with respect to μ

- Question is how much information we can get from these measures and what are the limitations?
- Back to the definition of $A(s) = \sum a(n)n^{-s}$, and let B is our original Dirichlet polynomial of arbitrary length.
- We first look at the first moment (the mean) i.e.

$$\frac{\int_T^{2T} B(\frac{1}{2}+it)|A(\frac{1}{2}+it)|^2 dt}{\int_T^{2T} |A(\frac{1}{2}+it)|^2 dt} = \int B(\tfrac{1}{2}+it) d\mu_A$$

In practice we use a smooth weight to close the bound of the integral.

- First moment (the mean) is calculabe as long as the length of A is smaller than $T^{1-\epsilon}$.

- First moment (the mean) is calculabe as long as the length of A is smaller than $T^{1-\epsilon}$.
- Unlike the Lebesgue measure the mean would carry useful information.

- First moment (the mean) is calculabe as long as the length of A is smaller than $T^{1-\epsilon}$.
- Unlike the Lebesgue measure the mean would carry useful information. What we describe next is often referred to as the Mollifier or resonance method in the context of analytic number theory.

- First moment (the mean) is calculabe as long as the length of A is smaller than $T^{1-\epsilon}$.
- Unlike the Lebesgue measure the mean would carry useful information. What we describe next is often referred to as the Mollifier or resonance method in the context of analytic number theory.
- Assume that we obtain

$$\int B(\tfrac{1}{2} + it) d\mu_A = \sigma,$$

then we can conclude that there exist $t \in [T, 2T]$ such that $B(\frac{1}{2} + it) > \sigma$.

- For example using the Mollifier method Soundararajan proved

- For example using the Mollifier method Soundararajan proved

Theorem (Soundararajan)

If T is sufficiently large then there exists $t \in [T, 2T]$ such that

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right).$$

- For example using the Mollifier method Soundararajan proved

Theorem (Soundararajan)

If T is sufficiently large then there exists $t \in [T, 2T]$ such that

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right).$$

Subsequently this been improved By Bondarenko and Seip to

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T \log\log\log T}}{\sqrt{\log\log T}}\right).$$

- For example using the Mollifier method Soundararajan proved

Theorem (Soundararajan)

If T is sufficiently large then there exists $t \in [T, 2T]$ such that

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right).$$

Subsequently this been improved By Bondarenko and Seip to

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T \log\log\log T}}{\sqrt{\log\log T}}\right).$$

- As we see calculating the mean is very useful.

- For example using the Mollifier method Soundararajan proved

Theorem (Soundararajan)

If T is sufficiently large then there exists $t \in [T, 2T]$ such that

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right).$$

Subsequently this been improved By Bondarenko and Seip to

$$\zeta(\tfrac{1}{2} + it) \geq \exp\left((1 + o(1))\frac{\sqrt{\log T \log\log\log T}}{\sqrt{\log\log T}}\right).$$

- As we see calculating the mean is very useful.
- What about the Variance?

The trivial bound

- Since we have that the mean equals to σ,

The trivial bound

- Since we have that the mean equals to σ, and by the fact that μ is a probability measure.

$$\int |B(\tfrac{1}{2}+it)|^2 d\mu_A > \sigma^2.$$

The trivial bound

- Since we have that the mean equals to σ, and by the fact that μ is a probability measure.

$$\int |B(\tfrac{1}{2}+it)|^2 d\mu_A > \sigma^2.$$

to see this

$$0 < \int |B - \sigma|^2 d\mu$$

The trivial bound

- Since we have that the mean equals to σ, and by the fact that μ is a probability measure.

$$\int |B(\tfrac{1}{2}+it)|^2 d\mu_A > \sigma^2.$$

to see this

$$0 < \int |B-\sigma|^2 d\mu = \int B^2 d\mu + \sigma^2 - 2\sigma \int B d\mu$$

The trivial bound

- Since we have that the mean equals to σ, and by the fact that μ is a probability measure.

$$\int |B(\tfrac{1}{2} + it)|^2 d\mu_A > \sigma^2.$$

to see this

$$0 < \int |B-\sigma|^2 d\mu = \int B^2 d\mu + \sigma^2 - 2\sigma \int B d\mu = -\sigma^2 + \int B^2 d\mu.$$

The trivial bound

- Since we have that the mean equals to σ, and by the fact that μ is a probability measure.

$$\int |B(\tfrac{1}{2} + it)|^2 d\mu_A > \sigma^2.$$

to see this

$$0 < \int |B - \sigma|^2 d\mu = \int B^2 d\mu + \sigma^2 - 2\sigma \int B d\mu = -\sigma^2 + \int B^2 d\mu.$$

$$\Rightarrow \sigma^2 < \int B^2.$$

Requirment for calculating variance

- Obviously understanding the variance gives a better understanding of our function.

Requirment for calculating variance

- Obviously understanding the variance gives a better understanding of our function. Question is, in which cases we can evaluate the variance?

Requirment for calculating variance

- Obviously understanding the variance gives a better understanding of our function. Question is, in which cases we can evaluate the variance?
- In general estimating the variance with respect to μ_A needs good knowledge of

$$\sum_{n<x} a(n)a(n+h), \quad \text{and} \sum_{n<x} a*b(n)a(n+h),$$

Requirment for calculating variance

- Obviously understanding the variance gives a better understanding of our function. Question is, in which cases we can evaluate the variance?
- In general estimating the variance with respect to μ_A needs good knowledge of

$$\sum_{n<x} a(n)a(n+h), \quad \text{and} \sum_{n<x} a*b(n)a(n+h),$$

which is a task that we need to find a way around it.

- In some cases we can beat the trivial bound we get from the mean.

- Let us dive into this with an L-series with important properties.

Definition

Let $\alpha > 0$ and define

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) := \frac{-2}{\alpha \log T} \sum_{n<T^\alpha} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}} \Big(1 - \frac{\log n}{\alpha \log T}\Big). \qquad (3)$$

- Let us dive into this with an L-series with important properties.

Definition

Let $\alpha > 0$ and define

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) := \frac{-2}{\alpha \log T} \sum_{n<T^\alpha} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}} \big(1 - \frac{\log n}{\alpha \log T}\big). \tag{3}$$

- Note that the length of $\mathcal{Z}_\alpha$ is T^α.
- For $\alpha < 1$ we have that

$$\int \mathcal{Z}_\alpha d\mu_{\text{Leb}} = 0.$$

- Let us dive into this with an L-series with important properties.

Definition

Let $\alpha > 0$ and define

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) := \frac{-2}{\alpha \log T} \sum_{n<T^\alpha} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}}\Big(1 - \frac{\log n}{\alpha \log T}\Big). \tag{3}$$

- Note that the length of $\mathcal{Z}_\alpha$ is T^α.
- For $\alpha < 1$ we have that

$$\int \mathcal{Z}_\alpha d\mu_{\text{Leb}} = 0.$$

- Again for $\alpha < 1$ the variance with respect to Lebesgue measure is

$$\int |\mathcal{Z}_\alpha|^2 d\mu_{\text{Leb}} = 1/3. \tag{4}$$

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$

For $\alpha = 1$, we set

$$\mathcal{Z}_1(\tfrac{1}{2}+it) = C(t) + i\, Im(t).$$

- We write
$$\mathcal{Z}_\alpha(\tfrac{1}{2} + it) = C_\alpha(t) + i\, Im_\alpha(t).$$
For $\alpha = 1$, we set
$$\mathcal{Z}_1(\tfrac{1}{2} + it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications.

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2} + it) = C_\alpha(t) + i\, Im_\alpha(t).$$

For $\alpha = 1$, we set

$$\mathcal{Z}_1(\tfrac{1}{2} + it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$

For $\alpha = 1$, we set

$$\mathcal{Z}_1(\tfrac{1}{2}+it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros.

- We write
$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$
For $\alpha = 1$, we set
$$\mathcal{Z}_1(\tfrac{1}{2}+it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros.

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2} + it) = C_\alpha(t) + i\, Im_\alpha(t).$$

For $\alpha = 1$, we set

$$\mathcal{Z}_1(\tfrac{1}{2} + it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros. Having $C(t) > 1$ is equivallent to having gaps smaller than half of the average between the zeros of the Riemann zeta function.

- We write
$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$
For $\alpha = 1$, we set
$$\mathcal{Z}_1(\tfrac{1}{2}+it) = C(t) + i\, Im(t).$$
- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros. Having $C(t) > 1$ is equivallent to having gaps smaller than half of the average between the zeros of the Riemann zeta function.
- Note that the support of the sum in the definition of $\mathcal{Z}_\alpha$ in on prime numbers.

- We write

$$\mathcal{Z}_\alpha(\tfrac{1}{2} + it) = C_\alpha(t) + i\, Im_\alpha(t).$$

For $\alpha = 1$, we set

$$\mathcal{Z}_1(\tfrac{1}{2} + it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros. Having $C(t) > 1$ is equivallent to having gaps smaller than half of the average between the zeros of the Riemann zeta function.
- Note that the support of the sum in the definition of $\mathcal{Z}_\alpha$ in on prime numbers. A heuristic suggests that $\mathcal{Z}_\alpha$ should behave similar to a sum of independent random variables. Hence, we expect its value to be normally dirtributed.

- We write
$$\mathcal{Z}_\alpha(\tfrac{1}{2}+it) = C_\alpha(t) + i\, Im_\alpha(t).$$
For $\alpha = 1$, we set
$$\mathcal{Z}_1(\tfrac{1}{2}+it) = C(t) + i\, Im(t).$$

- Understanding the behaviour of $\mathcal{Z}_\alpha$ has very important implications. For example proving there exist t such that $C(t) > 1$ or $C_2(t) > 1/2$, proves that there exist no Landau-Siegel zeros. Having $C(t) > 1$ is equivallent to having gaps smaller than half of the average between the zeros of the Riemann zeta function.
- Note that the support of the sum in the definition of $\mathcal{Z}_\alpha$ in on prime numbers. A heuristic suggests that $\mathcal{Z}_\alpha$ should behave similar to a sum of independent random variables. Hence, we expect its value to be normally dirtributed. One conclusion is (assuming the heuristc) $\mathcal{Z}_\alpha$ should get arbitray large.

- How much do we know about the real and imaginary parts of $\mathcal{Z}_\alpha$.

- How much do we know about the real and imaginary parts of $\mathcal{Z}_\alpha$. For the real part we have a neat expression relating it to the zeros of the Riemann zeta function.

- How much do we know about the real and imaginary parts of $\mathcal{Z}_\alpha$. For the real part we have a neat expression relating it to the zeros of the Riemann zeta function.
- Let γ denote an imaginary part of a zeta zero, and assume the Riemann hypothesis (RH) that predicts that all the non-trivial zeros are of the form $\frac{1}{2} + i\gamma$.

- How much do we know about the real and imaginary parts of $\mathcal{Z}_\alpha$. For the real part we have a neat expression relating it to the zeros of the Riemann zeta function.
- Let γ denote an imaginary part of a zeta zero, and assume the Riemann hypothesis (RH) that predicts that all the non-trivial zeros are of the form $\frac{1}{2} + i\gamma$. Then

$$C_\alpha(t) = \sum_{\zeta(\rho)=0} \left(\frac{\sin(\frac{\alpha}{2}(\gamma - t)\log T)}{\frac{\alpha}{2}(\gamma - t)\log T} \right)^2 - \alpha^{-1}, \qquad (5)$$

Which can be looked at as

$$C_\alpha(t) \simeq -\alpha^{-1} + \sum_{\text{if } t \text{ is close to } \gamma} 1.$$

- How much do we know about the real and imaginary parts of $\mathcal{Z}_\alpha$. For the real part we have a neat expression relating it to the zeros of the Riemann zeta function.
- Let γ denote an imaginary part of a zeta zero, and assume the Riemann hypothesis (RH) that predicts that all the non-trivial zeros are of the form $\frac{1}{2} + i\gamma$. Then

$$C_\alpha(t) = \sum_{\zeta(\rho)=0} \left(\frac{\sin(\frac{\alpha}{2}(\gamma - t)\log T)}{\frac{\alpha}{2}(\gamma - t)\log T} \right)^2 - \alpha^{-1}, \qquad (5)$$

Which can be looked at as

$$C_\alpha(t) \simeq -\alpha^{-1} + \sum_{\text{if } t \text{ is close to } \gamma} 1.$$

- Now you can see that if $C_\alpha(t)$ has a large value then we must have an accumulation of zeros in the vicinity of t.

Distribution of zeros of the zeta function.

- Now is a good place to say a few words about what we know and what is conjectured about the distribution of zeros.

Distribution of zeros of the zeta function.

- Now is a good place to say a few words about what we know and what is conjectured about the distribution of zeros.
- Montgomery in 1973 conjectured that the probability of having a gap smaller than β between zeros, is given by

$$\int_0^{\beta} 1 - \left(\frac{\sin(\pi u)}{\pi u}\right) du.$$

- His conjecture predicts that we have gaps arbitrary small,

Distribution of zeros of the zeta function.

- Now is a good place to say a few words about what we know and what is conjectured about the distribution of zeros.
- Montgomery in 1973 conjectured that the probability of having a gap smaller than β between zeros, is given by

$$\int_0^{\beta} 1 - (\frac{\sin(\pi u)}{\pi u}) du.$$

- His conjecture predicts that we have gaps arbitrary small, hence it rejects the existence of the Landau-Siegel zeros,

Distribution of zeros of the zeta function.

- Now is a good place to say a few words about what we know and what is conjectured about the distribution of zeros.
- Montgomery in 1973 conjectured that the probability of having a gap smaller than β between zeros, is given by

$$\int_0^{\beta} 1 - \left(\frac{\sin(\pi u)}{\pi u}\right) du.$$

- His conjecture predicts that we have gaps arbitrary small, hence it rejects the existence of the Landau-Siegel zeros, which was his motivation to study the distribution of gaps between zeta zeros.
- Under RH Montgomery and odlyzko proved there exist gaps smaller than 0.5179 times the average gap.

Distribution of zeros of the zeta function.

- Now is a good place to say a few words about what we know and what is conjectured about the distribution of zeros.
- Montgomery in 1973 conjectured that the probability of having a gap smaller than β between zeros, is given by

$$\int_0^\beta 1 - \left(\frac{\sin(\pi u)}{\pi u}\right) du.$$

- His conjecture predicts that we have gaps arbitrary small, hence it rejects the existence of the Landau-Siegel zeros, which was his motivation to study the distribution of gaps between zeta zeros.
- Under RH Montgomery and odlyzko proved there exist gaps smaller than 0.5179 times the average gap. Best is due to Preobrazhenski who showed existence of gaps smaller than 0.515396 times the average gaps. Reducing this to 0.5 is enough to reject the possibility of Landau-Siegel zeros.

Statements of results

- I already mentioned that large values of C_α implies the existence of small gaps.

Statements of results

- I already mentioned that large values of C_α implies the existence of small gaps. Also for $\alpha < 1$, we have that

$$\int C_\alpha^2(t) d\mu_{\text{Leb}} = 1/6 + O(1/\log T).$$

Statements of results

- I already mentioned that large values of C_α implies the existence of small gaps. Also for $\alpha < 1$, we have that

$$\int C_\alpha^2(t) d\mu_{\text{Leb}} = 1/6 + O(1/\log T).$$

  this shows that there exist t with positive Lebesgue measure, such that $C(t) > 0.4$.
- If we use other measure using the mollifier $A(s) = \sum \lambda(n) n^{-s}$, then we get

$$\int C(t) d\mu_\lambda = 0.666\cdots + O(1/\log T),$$

Statements of results

- I already mentioned that large values of C_α implies the existence of small gaps. Also for $\alpha < 1$, we have that

$$\int C_\alpha^2(t) d\mu_{\text{Leb}} = 1/6 + O(1/\log T).$$

  this shows that there exist t with positive Lebesgue measure, such that $C(t) > 0.4$.

- If we use other measure using the mollifier $A(s) = \sum \lambda(n) n^{-s}$, then we get

$$\int C(t) d\mu_\lambda = 0.666 \cdots + O(1/\log T),$$

  which gives that there exist t such that $C(t) > 0.666$.

Statements of results

- I already mentioned that large values of C_α implies the existence of small gaps. Also for $\alpha < 1$, we have that

$$\int C_\alpha^2(t) d\mu_{\text{Leb}} = 1/6 + O(1/\log T).$$

this shows that there exist t with positive Lebesgue measure, such that $C(t) > 0.4$.

- If we use other measure using the mollifier $A(s) = \sum \lambda(n) n^{-s}$, then we get

$$\int C(t) d\mu_\lambda = 0.666 \cdots + O(1/\log T),$$

which gives that there exist t such that $C(t) > 0.666$. This is already better than what we get from the Lebesgue measure.

- For the variance we prove

Corollary (A. 2020.)

Let $A(s) = \sum_{n<T^{1-\epsilon}} \lambda(n) n^{-s}$. *We have that*

$$\int C_\alpha^2(t) d\mu_\lambda > 0.4666.$$

- For the variance we prove

Corollary (A. 2020.)

Let $A(s) = \sum_{n<T^{1-\epsilon}} \lambda(n) n^{-s}$. We have that

$$\int C_\alpha^2(t) d\mu_\lambda > 0.4666.$$

Recall that the above is

$$\frac{1}{\log T} \int C_\alpha^2(t) \left| \sum_{n<T^{1-\epsilon}} \frac{\lambda(n)}{n^{1/2+it}} \right|^2 dt > 0.4666.$$

- This immediately shows that there exists t such $C(t) > 0.6831$, which is an improvement from the bound we get using the mean.

- For the variance we prove

Corollary (A. 2020.)

Let $A(s) = \sum_{n<T^{1-\epsilon}} \lambda(n) n^{-s}$. *We have that*

$$\int C_\alpha^2(t) d\mu_\lambda > 0.4666.$$

Recall that the above is

$$\frac{1}{\log T} \int C_\alpha^2(t) \left| \sum_{n<T^{1-\epsilon}} \frac{\lambda(n)}{n^{1/2+it}} \right|^2 dt > 0.4666.$$

- This immediately shows that there exists t such $C(t) > 0.6831$, which is an improvement from the bound we get using the mean. The "trivial" bound on the variance is $(2/3)^2 \simeq 0.444$.

- Now let us talk about the imaginary part of $\mathcal{Z}_\alpha$. The imaginary part Im_α is more mysterious due to its oscillatory nature.

- Now let us talk about the imaginary part of $\mathcal{Z}_\alpha$. The imaginary part Im_α is more mysterious due to its oscillatory nature. Also we do not have a formula that relates the imaginary part to the zeros directly.
- For the mean of the imaginary part we have

$$\int Im(t) d\mu_\lambda = 0.$$

- Now let us talk about the imaginary part of $\mathcal{Z}_\alpha$. The imaginary part Im_α is more mysterious due to its oscillatory nature. Also we do not have a formula that relates the imaginary part to the zeros directly.
- For the mean of the imaginary part we have

$$\int Im(t) d\mu_\lambda = 0.$$

- We can use the Landau-Gonek formula and obtain

$$\frac{\sum_{0<\gamma<T} Im_{1-\epsilon}(\gamma + \frac{2\pi d}{\log T})}{\sum_{0<\gamma<T} 1} \sim \int_0^1 2u(1-u)\sin(2\pi u d) du.$$

- Now let us talk about the imaginary part of $\mathcal{Z}_\alpha$. The imaginary part Im_α is more mysterious due to its oscillatory nature. Also we do not have a formula that relates the imaginary part to the zeros directly.
- For the mean of the imaginary part we have

$$\int Im(t)d\mu_\lambda = 0.$$

- We can use the Landau-Gonek formula and obtain

$$\frac{\sum_{0<\gamma<T} Im_{1-\epsilon}(\gamma + \frac{2\pi d}{\log T})}{\sum_{0<\gamma<T} 1} \sim \int_0^1 2u(1-u)\sin(2\pi ud)du.$$

The RHS is zero for $d = 0$, but if we set $d \simeq 0.41$, we get 0.27 for the RHS,

- Now let us talk about the imaginary part of $\mathcal{Z}_\alpha$. The imaginary part Im_α is more mysterious due to its oscillatory nature. Also we do not have a formula that relates the imaginary part to the zeros directly.
- For the mean of the imaginary part we have

$$\int Im(t) d\mu_\lambda = 0.$$

- We can use the Landau-Gonek formula and obtain

$$\frac{\sum_{0<\gamma<T} Im_{1-\epsilon}(\gamma + \frac{2\pi d}{\log T})}{\sum_{0<\gamma<T} 1} \sim \int_0^1 2u(1-u)\sin(2\pi u d)du.$$

The RHS is zero for $d = 0$, but if we set $d \simeq 0.41$, we get 0.27 for the RHS, which is the max.

- Imaginary part is important due to its tie to the famous $S(t)$ function.
- $S(t)$ is the deviation from the expected value in the zero counting function of the Riemann zeta function.

- Imaginary part is important due to its tie to the famous $S(t)$ function.
- $S(t)$ is the deviation from the expected value in the zero counting function of the Riemann zeta function.

$$S(t) = \sum_{0<\gamma<T} 1 - \frac{T}{2\pi} \log\Big(\frac{T}{2\pi e}\Big) - \frac{7}{8} + O\Big(\frac{1}{T}\Big).$$

- Selberg proved that

Theorem (Selberg)

For $k \geq 1$, we have

$$\int_0^T \left| S(t) + \frac{1}{\pi} \Im \sum_{n<T^{1/k}} \frac{\Lambda(n)}{n^{\frac{1}{2}+it} \log n} \right|^k \ll_k T.$$

- The theorem shows that, on average, the series expansion on the RHS is approximating $S(t)$.

- Using the theorem, Selberg showed that all even moments of $S(t)$ match the moments of the normal distribution, which confirm the speculation that $S(t)$ should behave like a sum of independent RV.

- Using the theorem, Selberg showed that all even moments of $S(t)$ match the moments of the normal distribution, which confirm the speculation that $S(t)$ should behave like a sum of independent RV.
- As we proceed, I will give other important properties of the imaginary.

- Using the theorem, Selberg showed that all even moments of $S(t)$ match the moments of the normal distribution, which confirm the speculation that $S(t)$ should behave like a sum of independent RV.
- As we proceed, I will give other important properties of the imaginary. But first let show the results we get for the imaginary part.

Corollary (A. 2020.)

We have that

$$\int Im^2(t) d\mu_\lambda > 0.1.$$

An interesting arithmetic function

- This shows that we can find t that $Im(t) > 0.3162.$, which is an improvement to 0.27.
- Let us introduce a somewhat genarlization of the Liouville function. Let $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, and define

$$\lambda_2(n) = (-1)^{[\frac{\Omega(n)}{2}]},$$

where $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, and $[\cdot]$ is the floor function.

An interesting arithmetic function

- This shows that we can find t that $Im(t) > 0.3162.$, which is an improvement to 0.27.
- Let us introduce a somewhat genarlization of the Liouville function. Let $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, and define

$$\lambda_2(n) = (-1)^{[\frac{\Omega(n)}{2}]},$$

where $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, and $[\cdot]$ is the floor function.

Example

We have $\lambda_2(1) = \lambda_2(p_1) = 1, \ \ \lambda_2(p_1p_2) = \lambda_2(p_1p_2p_3) = -1,$ $\lambda_2(p_1p_2p_3p_4) = 1.$

- The L-function defined using λ_2,

$$L(s,\lambda_2)=\sum_{n=1}^{\infty}\frac{\lambda_2(n)}{n^s},$$

is mysterious.

- The L-function defined using λ_2,

$$L(s,\lambda_2)=\sum_{n=1}^{\infty}\frac{\lambda_2(n)}{n^s},$$

is mysterious.

- This is unlike the L-function defined using the Liouville function.

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\frac{\zeta(2s)}{\zeta(s)}.$$

- The L-function defined using λ_2,

$$L(s, \lambda_2) = \sum_{n=1}^{\infty} \frac{\lambda_2(n)}{n^s},$$

is mysterious.

- This is unlike the L-function defined using the Liouville function.

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}.$$

- We use λ_2 as for coefficients of our mollifier and we obtain the following result.

Corollary (A. 2020)

We have that

$$\int Im^2(t) d\mu_{\lambda_2} > 0.4666.$$

- This shows that μ_{λ_2} has values distributed where imaginary part has large values.

Corollary (A. 2020)

We have that

$$\int Im^2(t)d\mu_{\lambda_2} > 0.4666.$$

- This shows that μ_{λ_2} has values distributed where imaginary part has large values.
- Also the corollary shows that there exist t such that

$$Im(t) > 0.6831.$$

Corollary (A. 2020)

We have that

$$\int Im^2(t) d\mu_{\lambda_2} > 0.4666.$$

- This shows that μ_{λ_2} has values distributed where imaginary part has large values.
- Also the corollary shows that there exist t such that

$$Im(t) > 0.6831.$$

- Now let us relate the real and the imaginary parts together.

Corollary (A. 2020)

We have that

$$\int \Big(C^2(t) - Im^2(t)\Big) d\mu_\lambda = 0.36666\cdots + O\big(\tfrac{1}{\log T}\big),$$
$$\int \Big(C^3(t) - 3C(t)Im^2(t)\Big) d\mu_\lambda = 0.16504\cdots + O\big(\tfrac{1}{\log T}\big),$$
$$\int \Big(C^4(t) - 6C^2(t)Im^2(t) + Im^4(t)\Big) d\mu_\lambda = 0.06194\cdots + O\big(\tfrac{1}{\log T}\big).$$

Corollary (A. 2020)

We have that

$$\int \Big(C^2(t) - Im^2(t)\Big) d\mu_\lambda = 0.36666 \cdots + O\big(\tfrac{1}{\log T}\big),$$

$$\int \Big(C^3(t) - 3C(t)Im^2(t)\Big) d\mu_\lambda = 0.16504 \cdots + O\big(\tfrac{1}{\log T}\big),$$

$$\int \Big(C^4(t) - 6C^2(t)Im^2(t) + Im^4(t)\Big) d\mu_\lambda = 0.06194 \cdots + O\big(\tfrac{1}{\log T}\big).$$

- As I mentioned, Important problem is wheather or not there exist t such that $C(t) > 1$.

Corollary (A. 2020)

We have that

$$\int \Big(C^2(t) - Im^2(t)\Big) d\mu_\lambda = 0.36666 \cdots + O\big(\tfrac{1}{\log T}\big),$$
$$\int \Big(C^3(t) - 3C(t)Im^2(t)\Big) d\mu_\lambda = 0.16504 \cdots + O\big(\tfrac{1}{\log T}\big),$$
$$\int \Big(C^4(t) - 6C^2(t)Im^2(t) + Im^4(t)\Big) d\mu_\lambda = 0.06194 \cdots + O\big(\tfrac{1}{\log T}\big).$$

- As I mentioned, Important problem is wheather or not there exist t such that $C(t) > 1$.
- Now the above result shows real and imaginary parts are intertwined and a lower bounds on one of them can be used to get a lower bounds on the other.

- If we accept the heuristic or the pair correlation conjecture higher moments of the real part should explode:

$$\int C^k(t) d\mu_\lambda \to \infty \text{ as } k \to \infty.$$

- We get the above results from a more general theorem.

- If we accept the heuristic or the pair correlation conjecture higher moments of the real part should explode:

$$\int C^k(t)d\mu_\lambda \to \infty \text{ as } k \to \infty.$$

- We get the above results from a more general theorem.

Theorem (A. 2020)

$$\int \sum_{n=0}^{[k/2]} \binom{k}{2n} (-1)^n C^{k-2n}(t) Im^{2n}(t) d\mu_\lambda$$

$$= 2^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{1}{(k+i+1)!} + O(\frac{1}{\log T}).$$

- The RHS of the theorem is approximately

$$\frac{2^k}{(k+1)!}.$$

- A question is how much information we can derive from the theorem?

- The RHS of the theorem is approximately

$$\frac{2^k}{(k+1)!}.$$

- A question is how much information we can derive from the theorem? Would the theorem imply that $C(t)$, is bounded,

- The RHS of the theorem is approximately

$$\frac{2^k}{(k+1)!}.$$

- A question is how much information we can derive from the theorem? Would the theorem imply that $C(t)$, is bounded, or can we find examples that it remains unbounded.
- Next let us give a relevant example.

Example

Let $Z := XU$, where X and U are independent with $X > 0$.

Example

Let $Z := XU$, where X and U are independent with $X > 0$.
Assume that X is unbounded with $E[X^k] < \infty$.

Example

Let $Z := XU$, where X and U are independent with $X > 0$. Assume that X is unbounded with $E[X^k] < \infty$. Let $U = e^{iT}$ where T is a RV with values in the interval $[0, 2\pi)$ and its pdf is given by

$$p(t) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} a_n \cos nt\right).$$

Example

Let $Z := XU$, where X and U are independent with $X > 0$. Assume that X is unbounded with $E[X^k] < \infty$. Let $U = e^{iT}$ where T is a RV with values in the interval $[0, 2\pi)$ and its pdf is given by

$$p(t) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} a_n \cos nt\right).$$

Now for $k \in \mathbb{N}$ we have

$$E[U^k] = E[e^{ikT}] = \int_0^{2\pi} e^{ikt} p(t) dt = a_k,$$

Example

Let $Z := XU$, where X and U are independent with $X > 0$. Assume that X is unbounded with $E[X^k] < \infty$. Let $U = e^{iT}$ where T is a RV with values in the interval $[0, 2\pi)$ and its pdf is given by

$$p(t) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} a_n \cos nt\right).$$

Now for $k \in \mathbb{N}$ we have

$$E[U^k] = E[e^{ikT}] = \int_0^{2\pi} e^{ikt} p(t) dt = a_k,$$

which by letting $0 < a_k \sim \frac{2^k}{(k+1)!E[X]^k}$, and by the independence of X, U we have that

$$E[Z^k] = E[(XU)^k] \sim \frac{2^k}{(k+1)!}.$$

Explaination of the proof

- Let $\sigma = \frac{1}{2} + \frac{\beta}{\log T}$ where $\beta \gg 1$. Assuming the Riemann hypothesis Selberg proved

$$\frac{1}{T}\int_0^T \left|\frac{1}{\log T}\frac{\zeta'}{\zeta}(\sigma + it)\right|^2 dt \ll_\beta 1. \tag{6}$$

Explaination of the proof

- Let $\sigma = \frac{1}{2} + \frac{\beta}{\log T}$ where $\beta \gg 1$. Assuming the Riemann hypothesis Selberg proved

$$\frac{1}{T}\int_0^T \left|\frac{1}{\log T}\frac{\zeta'}{\zeta}(\sigma + it)\right|^2 dt \ll_\beta 1. \tag{6}$$

- Later Goldston, Gonek and Montgomery proved that getting an asymptotic for (6) is equivalent to Montgomery's pair correlation conjecture on the distribution of the spacing between zeta zeros.

Explaination of the proof

- Let $\sigma = \frac{1}{2} + \frac{\beta}{\log T}$ where $\beta \gg 1$. Assuming the Riemann hypothesis Selberg proved

$$\frac{1}{T}\int_0^T \left|\frac{1}{\log T}\frac{\zeta'}{\zeta}(\sigma + it)\right|^2 dt \ll_\beta 1. \tag{6}$$

- Later Goldston, Gonek and Montgomery proved that getting an asymptotic for (6) is equivalent to Montgomery's pair correlation conjecture on the distribution of the spacing between zeta zeros.
- In the proof they use the simple fact that

$$\left|\frac{\zeta'}{\zeta}(s)\right|^2 = 2\Re\Big(\frac{\zeta'}{\zeta}(s)\Big)^2 - \Re\Big(\big(\frac{\zeta'}{\zeta}(s)\big)^2\Big). \tag{7}$$

Then they integrate the LHS of (7) with respect to the Lebesgue measure.

Consequently the last term in RHS of the (7) disappears and only

$$\Re\left(\frac{\zeta'}{\zeta}(s)\right)^2$$

plays a role. As we mentioned the real part is closely tied to the distribution of zeros. Therefore, they used Montgomery's study of the pair correlation to get their results.

Consequently the last term in RHS of the (7) disappears and only

$$\Re\Big(\frac{\zeta'}{\zeta}(s)\Big)^2$$

plays a role. As we mentioned the real part is closely tied to the distribution of zeros. Therefore, they used Montgomery's study of the pair correlation to get their results. In our work by switching the measure from the Lebesgue, we take most advantage of

$$\Re\Big(\big(\frac{\zeta'}{\zeta}(s)\big)^2\Big),$$

in the RHS of (7), which does not contribute when we use the Lebesgue measure.

Thank you!