Zeta function for moduli G-Shtukas

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(joint work with Tuan Ngo Dac (in progress...)
(joint works with Urs Hartl))

March 25, 2021

Outline Of The Talk

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- Beginning

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- Beginning
- Middle

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- Beginning
- Middle
- End

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"... but there is not enough space in the margin..."

§ acknowledgement

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- Organizers

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- Can Ozan

§ Motives

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- - standard conjectures) Mot (k) is abelian semi-simple.
- - (U. Jannsen's semisimplicity result 1992) k = F_q and = num
  then Mot (k) is abelian semisimple

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- - Cisinski and Deglise: $DM(X)$ over general base scheme X (2010)

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There are certain candidates for the true category of motives over function fields.

- Category of t-motives (Anderson 1986)
  Def: $A := F_q[t]$, L an A-field via $A \,!\, L;\ t \,7!$ (char. morphism). An effective t-motive of rk r over L is a pair $\underline{M} = (M;\ )$
  - a free and f.g. A_L-module M of rank r, and
  - $:\ M := M \ _{A_L;}\ A_L \,!\, M$ (s. th. $(t \quad)^d$ annihilates coker).

  Here $: A_L \,!\, A_L;\ a \quad b \;7!\; a \quad b^q$.

- Taelman's category M (Dissertation 200?)

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  - Hom(;) Q (Q = Frac(A))
  - formally invert (tensor powers of) the Carlitz motive $\underline{C}$.

$$\underline{C} \text{ Carlitz motive!} \quad M(G_m)$$

The resulting category tM together with the obvious ber functor ! : tM ! Q vector spaces provides a tannakian category which is a candidate for the analogous motivic category over function elds. Still one may naturally want:

- multiplication by a Dedekind domain which is strictly bigger than $F_q[t]$,

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1. replace t-motives with A-motives, make A-motives completed at the place in nity 1 of a curve C and replace M by vector bundle $M = C_L$

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2. Let's invert further Carlitz(-Hayes) motives (this corresponds to introducing further characteristic sections $s : \mathrm{Spec}\, L \to C$)

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2. Let's invert further Carlitz(-Hayes) motives (this corresponds to introducing further characteristic sections $s : \operatorname{Spec} L \to C$)

One can then easily see that the resulting category is equivalent with the following category

§Cat Of C-Motives

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De nition

Let C be a sm. proj. curve over $\mathbb{F}_q$. Fix $\underline{\ }$:= ($_i$) 2 C^n. Let S 2 Sch$=\mathbb{F}_q$. A C-motive $\underline{M}\_$ with char _ over S is a tuple (M ; $_M$)

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- a loc. free sheaf M of $\mathcal{O}_{C_S}$-mod of finite rk,

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- a loc. free sheaf M of $\mathcal{O}_{C_S}$-mod of finite rk,
- $\ _M : \ M|_{\ } \xrightarrow{\sim} M|_{\ }$ where $M|_{\ }$ is the restriction of M to C_S $(C_{\ } = C \setminus \underline{\ })$, and $\ = id \ _S$ where $\ _S : S \to S$ is the abs. Frob$/\mathbb{F}_q$.

The set of quasi-morphisms $QHom(\underline{M}, \underline{N})$ is given by

$$\begin{array}{ccc} M & \xrightarrow{\ _M} & M \\ \Big| & & \Big| \\ N & \xrightarrow{\ _N} & N \end{array}$$

We denote the resulting category by $\underline{M}ot_{\bar{C}}(S)$.

Theorem (Analog of Jannsen's semisimplicity result)

The category $Mot_{\overline{C}}(\overline{F}_q)$ with the obvious ber functor! is a semi-simple tannakian category. In particular the associated motivic group P is pro-reductive.

Proof.

cf. [Ara-Har I] □

Remark

For this category one can establish

-realization functiors

-Tate conjecture

-analog for Honda-Tate theory (there is a function elds Weil number pro-torus) and etc... [Ara-Har I]

§G-shtukas as G-motives

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Still:

-Want to geometrize this cat!

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§G-shtukas as G-motives

Still:

-Want to geometrize this cat!

-Want to equip $Mot_{\bar{C}}(S)$ with $\mathfrak{G}$-structure for an affine flat group scheme over C of f.t.!

Recall:

$H^1(C; \mathfrak{G})(S)$ = Cat of $\mathfrak{G}$-bundles over C_S

Theorem

Let $\mathfrak{G}$ be a flat affine group scheme of finite type over the curve C. Then the stack $H^1(C; \mathfrak{G})$ is an Artin-stack locally of finite type over $\mathbb{F}_q$. It admits a covering by connected open substacks of finite type over $\mathbb{F}_q$. If $\mathfrak{G}$ is smooth over C then $H^1(C; \mathfrak{G})$ is smooth over $\mathbb{F}_q$.

Proof.

cf. [Ara-Har II] Theorem 2.5. □

Definition (Global G-shtuka)

A global G-shtuka $\underline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \underline{s}, \varphi)$ consisting of

- a G-bundle $\mathcal{G}$ over C_S,
- an n-tuple $\underline{s} := (s_i) \in C^n(S)$ of (characteristic) sections and
- an isomorphism $\varphi : \sigma^*\mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}} \xrightarrow{\sim} \mathcal{G}|_{C_S \setminus \Gamma_{\underline{s}}}$.

We let $\nabla_n H^1(C, G)$ denote the stack whose S-points parameterizes global G-shtukas over S.

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We let $\nabla_n H^1(C, G)$ denote the stack whose S-points parameterizes global G-shtukas over S.

Theorem

$\nabla_n H^1(C, G)$ is an ind-DM-stack over C^n which is ind-separated and locally of ind-finite type.

Proof.

cf. [Ara-Har II; theorem 3.15] □

Remark (Functoriality)

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The assignment

$$(C;G) \mapsto r \quad {}_nH^1(C;G)$$

is functorial on $(C;G)$. In particular $: G \to \quad G'$ induces

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which is induced by

$$: H^1(C, G) \to H^1(C, G'); \; G \mapsto G \times^G G':$$

between the stack of G-bundles and G'-bundles over C.

This allows to think of $r_n H^1(C; G)$ as a moduli for motives with G-structure

$$\underline{G} \quad \varphi_{\underline{G}} : \mathrm{Rep}\,G \to \mathcal{M}ot_{\overline{C}}(S)$$

§Realization Functors I

De nition (Etale realization)

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Fix $\xi = (\xi_i) \in \mathbb{C}^n$. Let A_ξ denote the completion of $\mathcal{O}_{\mathbb{C}^n}$ at ξ and let $r\,{}_nH^1(C;G)_\xi = r\,{}_nH^1(C;G) \otimes_{\mathbb{C}^n} \mathrm{Spf}\,A_\xi$. Assume that S is connected, fix a geometric base point $\bar{s}$ of S.

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Fix _ = (i) 2 C^n. Let A_ denote the completion of O$_{C^n}$ at _ and let r $_n$H^1(C; G)_ = r $_n$H^1(C; G) $_{C^n}$ Spf A_. Assume that S is connected, x a geometric base point s̄ of S. There is the following etale realization functor

$$! _( ) : r _nH^1(C; G)_(S) ! \quad Funct \quad Rep(G; Mod_{A_[^1(S;\bar{s})]}$$

$$\underline{G} \; 7! \; ! _(\underline{G}) : \quad 7! \; \lim_{D \; C_} (\quad G j_{D_{\bar{s}}}) \quad _{O_} A_$$

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Fix _ = ($_i$) 2 C^n. Let $A_{\_}$ denote the completion of O_{C^n} at _ and let r $_n H^1(C;G)_{\_}$ = r $_n H^1(C;G)$ $_{C^n}$ Spf $A_{\_}$. Assume that S is connected, x a geometric base point $\bar{s}$ of S. There is the following etale realization functor

$$! _{\_}(\ ) : r \ _n H^1(C;G)_{\_}(S) \ ! \quad \text{Funct} \quad \text{Rep}(G; \text{Mod}_{A_{\_}[\ ^1(S;\bar{s})]}$$

$$\underline{G} \ 7! \ !_{\_}(\underline{G}) : \quad 7! \ \lim_{D \ C_{\_}} (\quad \underline{G}|_{D_{\bar{s}}}) \quad {}_{O_{\_}} A_{\_}$$

Here - $_1(S;s)$ is the algebraic fundamental group of S.
-$D_{\bar{s}}$ is nite over $\bar{s}$ = Spec F for an algebraically closed eld F, and
$\underline{G}|_{D_{\bar{s}}}$ is equivalent to (M;) where M is a free $O_{D_{\bar{s}}}$-modules.
Then ($\underline{G}|_{D_{\bar{s}}}$) := f m 2 M : (m) = mg denotes the -invariant

§Level Structure

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Definition (H-level structure)

Assume that $S \in Nilp_{A_\_}$ is connected and $\bar{s}$ a geometric point of S. For a global G-shtuka $\underline{\mathcal{G}}$ over S let us consider the set of isomorphisms of tensor functors

$$\mathrm{Isom}^{\otimes}(\omega^{-}(\underline{\mathcal{G}})(\;),\; \omega^{\circ}(\;));$$

where $\omega^{\circ} : Rep_{A_\_}\, G \to Mod_{A_\_}$ denote the neutral fiber functor.

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De nition (H-level structure)

Assume that S 2 N ilp$_{A_}$ is connected and x a geometric point s of S. For a global G-shtuka $\underline{G}$ over S let us consider the set of isomorphisms of tensor functors

$$\text{Isom}\ (!\ \_(\underline{G})(\ \ );!\ \ (\ \ ));$$

where ! : Rep$_{A_}$ G ! Mod$_{A_}$ denote the neutral ber functor. The set Isom (! $\_(\underline{G})(\ \ );!\ \ (\ \ ))$ admits an action of G(A$_$) $_1$(S; s) where G(A$_$) acts through ! ( ) by tannakian formalism and $_1$(S; s) acts through ! $\_(\underline{G})(\ \ )$. For a compact open subgroup H G(A$_$) we de ne a rational H-level structure on a global G-shtuka $\underline{G}$ over S 2 N ilp$_{A_}$ to be a $_1$(S; s)-invariant H-orbit = H in Isom (! $\_(\underline{G})(\ \ );!\ \ (\ \ ))$.

De nition (Loop groups/ a ne ag varieties)

§Realization Functors II

Crystalline realizations

Definition (Loop groups/ affine flag varieties)

Let $\mathcal{P}$ be a smooth affine group scheme of finite type over $D = \operatorname{Spec} k[\![z]\!]$, P the generic fiber of $\mathcal{P}$ over $\dot{D} := \operatorname{Spec} k(\!(z)\!)$.

De nition (Loop groups/ a ne ag varieties)

Let P be a smooth a ne group scheme of nite type over $D = \mathrm{Spec}\, k[[z]]$, P the generic ber of P over $\dot{D} := \mathrm{Spec}\, k((z))$.

1. The group of positive loops (resp. loops) associated with P

$$L^+P(R) := P(R[[z]]) := P(D_R) := \mathrm{Hom}_D(D_R; P);$$

$$(\text{resp. } LP(R) := P(R((z))) := P(\dot{D}_R) := \mathrm{Hom}_{\dot{D}}(\dot{D}_R; P));$$

where we write $R((z)) := R[[z]][\frac{1}{z}]$ and $\dot{D}_R := \mathrm{Spec}\, R((z))$. It is representable by a scheme (resp. an ind-scheme) of nite type (resp. ind- nite type) over k.

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$$L^+\mathcal{P}(R) := \mathcal{P}(R[\![z]\!]) := \mathcal{P}(D_R) := \operatorname{Hom}_D(D_R; \mathcal{P})\ ;$$

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   where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{D}_R := \operatorname{Spec} R(\!(z)\!)$. It is representable by a scheme (resp. an ind-scheme) of nite type (resp. ind- nite type) over k.
2. The a ne ag variety $\mathcal{F}\ell_{\mathcal{P}}$ is de ned to be the ind-scheme representing the fpqc-sheaf associated with the presheaf

$$R \mapsto LP(R)/L^+\mathcal{P}(R) = P(R(\!(z)\!))/\mathcal{P}(R[\![z]\!]):$$

   on the category of k-algebras

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- an isomorphism of the associated loop group torsors $\hat{\ } \colon \hat{\ } L \to L$.

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a) A local P-shtuka over $S \in Nilp_{k[\![\ ]\!]}$ is a pair $\underline{L} = (L_+, \hat{\tau})$ consisting of

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b) A morphism (quasi-isogeny) between $\underline{L} := (L_+, \tau) \to \underline{L}' := (L'_+, \tau')$ is a commutative diagram

$$\begin{array}{ccc} \hat{\sigma}^* L & \xrightarrow{\tau} & L \\ \downarrow & & \downarrow \\ \hat{\sigma}^* L' & \xrightarrow{\tau'} & L' \end{array}$$

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$$\begin{array}{ccc} L & \longrightarrow & L \\ \big| & & \big| \\ \hat{\ } L' & \xrightarrow{\ '} & L' \end{array}$$

c) We denote the resulting category by $LocP\text{-}Sht(S)$.

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There is a crystalline realization functor

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given by sending G to its formal completion $\hat{G}$ along ${}_{s_i}$ C_S and then using the following observation

$$\text{Cat of formal } \hat{P}\text{-torsors}/D_R \ \$\ \ \text{Cat of } L^+ P\text{-torsors}$$

Here $\hat{P}$ is the completion of P at $V(z)$.

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Let P be a smooth affine group scheme over $\mathcal{O}$.

- a) A closed ind-subscheme $\hat{Z}$ of $\widehat{\mathcal{F}\ell}_P := \mathcal{F}\ell_P \hat{\times}_k \operatorname{Spf} k[[\zeta]]$ which is stable under the left L^+P-action, such that $Z := \hat{Z} \hat{\times}_{\operatorname{Spf} k[[\zeta]]} \operatorname{Spec} k$ is a quasi-compact subscheme of $\mathcal{F}\ell_P$ is called a bound.

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- a) A closed ind-subscheme $\hat{Z}$ of $\widehat{F\ell}_P := F\ell_P \hat{\times}_k \operatorname{Spf} k⟦\zeta⟧$ which is stable under the left L^+P-action, such that $Z := \hat{Z} \hat{\times}_{\operatorname{Spf} k⟦\zeta⟧} \operatorname{Spec} k$ is a quasi-compact subscheme of $F\ell_P$ is called a bound.

- b) Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}}$. Let $\mathcal{L}_+$ and $\mathcal{L}_+'$ be L^+P-torsors over a scheme S in $\mathcal{N}ilp_{R_{\hat{Z}}}$ and let $\delta : \mathcal{L} \to \mathcal{L}'$ be an isomorphism of the associated LP-torsors. We consider an étale covering $S' \to S$ over which trivializations $\alpha : \mathcal{L}_+ \to (L^+P)_{S'}$ and $\alpha' : \mathcal{L}_+' \to (L^+P)_{S'}$ exist. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1}$ of $(LP)_{S'}$ corresponds to a morphism $S' \to LP \hat{\times}_k \operatorname{Spf} R_{\hat{Z}}$. We say that δ is bounded by $\hat{Z}$ if for any such trivialization and for all finite extensions R of $k⟦\zeta⟧$ over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists the induced morphism

$$S^0 \to \widehat{F}_{\ell_P}$$

factors through $\widehat{Z}_R$.

$$S' \to \widehat{\mathcal{F}\ell}_P$$

factors through $\hat{Z}_R$.

c) a local P-shtuka $(\mathcal{L}, \hat{\tau})$ is bounded by $\hat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\hat{Z}$. Assume that $\hat{Z} = S(\omega) \hat{\times}_k \mathrm{Spf}\, k[[\zeta]]$ for a Schubert variety $S(\omega) \subset \mathcal{F}\ell_P$, with $\omega \in \widetilde{W}$. Then we say that $\hat{\tau}$ is bounded by ω.

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c) a local $\mathbb{P}$-shtuka $(\mathcal{L}, \hat{\tau})$ is bounded by $\hat{Z}$ if the isom $\hat{\tau}^{-1}$ is bounded by $\hat{Z}$. Assume that $\hat{Z} = S(\omega) \hat{\times}_k \operatorname{Spf} k[[\;]]$ for a Schubert variety $S(\omega) \subset \mathcal{F}\ell_P$, with $\omega \in \tilde{W}$. Then we say that is bounded by ω.

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Remark

1. The above definition is a naive definition of BC. For the true definition one needs to replace $\hat{Z}$ with an equivalence class $[\hat{Z}]$ of subschemes of $\widehat{\mathcal{F}\ell}_{P,R}$. Here R is a finite extension of discrete valuation rings $k[[\zeta]] \subset R \subset k((\zeta))^{alg}$. The class $[\hat{Z}]$ has a representative over a minimal ring $R_{[\hat{Z}]}$ (called reflex ring)

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1. The above definition is a naive definition of BC. For the true definition one needs to replace $\hat{Z}$ with an equivalence class $[\hat{Z}]$ of subschemes of $\hat{F}\ell_{P,R}$. Here R is a finite extension of discrete valuation rings $k[[\ ]] \subset R \subset k((\ ))^{alg}$. The class $[\hat{Z}]$ has a representative over a minimal ring $R_{[\hat{Z}]}$ (called reflex ring)
2. There is a global version of the BC, which we obtain roughly by replacing $F\ell_P$ by B-D affine Grassmannian $GR_n(C; G)$, and $\hat{Z} \subset \hat{F}\ell_{P,R}$ by global Schubert varieties $Z \subset GR_n(C; G)$. Then BC $[Z]$ determines a minimal curve of definition C_Z called reflex curve.

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-a compact open subgroup H $G(A_{\overline{C}})$.
b) There is a functorial assignment

$$(G; \hat{\underline{Z}}; H) \qquad r\,{}^{\hat{Z};H}_{n} H^1(C; G)$$

where $r\,{}^{\hat{Z};H}_{n} H^1(C; G)(S)$ parametrizes $(\underline{G};\ )$ such that $!\ _i(\underline{G})$ is bounded by $\hat{Z}_i$.

Theorem

a) $r^{\frac{\hat{Z}}{n};H} H^1(C; G)$ is a formal DM-stack over $R_{\hat{Z}}$.

b) For H small enough $r^{\frac{\hat{Z}}{n};H} H^1(C; G)$ is quasi-projective.

Proof.

cf. [Ara-Har I]

$\square$

Recall:

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- X proper smooth over $\mathbb{Q}$ with good reduction at p

$$\log \zeta(X; s) = \sum_r \# X(\mathbb{F}_{p^r}) p^{-rs}/r$$

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In general,

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Q: Describe

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Q1 Compute

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Definition (Local rH -data)

A local rH -datum is a tuple $(P; \hat{Z}; b)$ consisting of

- A smooth affine group scheme P over D with connected reductive generic fiber P,
- A local bound $\hat{Z}$,
- A -conjuagacy class of an element $b \in P(\overline{k}((z)))$.

Definition (R-Z spaces for local P-shtukas)

Let $\hat{Z} = [\hat{Z}_R]$ be a bound with reflex ring $R_{\hat{Z}}$.
Fix a local P-shtuka $\underline{L}$ over k.

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Define the Rapoport-Zink space for (bounded) local P-shtukas, as the space given by the following functor of points

$$\underline{M}^{\hat{Z}}_{\underline{L}} : (Nilp_{R_{\hat{Z}}})^o \to Sets$$

$$S \mapsto \text{Isomorphism classes of } (\underline{L}, \bar{\delta}) : \text{ where:}$$

$\underline{L}$ is a local P-shtuka over S bounded by $\hat{Z}$ and $\bar{\delta} : \underline{L}_{\bar{S}} \to \underline{L}_{\bar{S}}$ a quasi-isogeny.

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& S \mapsto \text{Isomorphism classes of } (\underline{L}, \ ) : \text{ where:} \\
& \qquad \underline{L} \text{ is a local P-shtuka} \\
& \qquad \text{over } S \text{ bounded by } \hat{Z} \text{ and} \\
& \qquad \bar{\ } : \underline{L}_{\bar{S}} \to \underline{L}_{\bar{S}} \text{ a quasi-isogeny.}
\end{aligned}
$$

Here $\bar{S} := V(\ ) \quad S$.

Theorem (Representablity Of R-Z spaces for local $\mathbb{P}$-shtukas)

The functor $\underline{\mathcal{M}}_{\underline{L}}^{\hat{Z}}$ is ind-representable by a formal scheme over $\operatorname{Spf} R_{\hat{Z}}$ which is locally formally of finite type and separated. It is an ind-closed ind-subscheme of $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}_q} \operatorname{Spf} R_{\hat{Z}}$. Its underlying reduced subscheme equals a closed ADLV $X_Z(b)$, which is a scheme locally of finite type and separated over $\mathbb{F}$, all of whose irreducible components are projective.

Proof.

cf. [Ara-HarIV], Theorem 4.18. □

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Definition

The datum $(\mathbb{P}, \hat{Z}, b)$ determines the reflex ring $R_{\hat{Z}}$, and a local P-shtuka $\underline{L} := (L^+\mathbb{P}, b\hat{\sigma})$. This establishes

$$(\mathbb{P}, \hat{Z}, b) \mapsto \mathcal{M}(\mathbb{P}, \hat{Z}, b) := \mathcal{M}_{\underline{L}}^{\hat{Z}}, \qquad (1)$$

- which assigns the Rapoport-Zink space $\mathcal{M}(\mathbb{P}, \hat{Z}, b) := \underline{\mathcal{M}}_{\underline{L}}^{\hat{Z}}$ to a local $\nabla\mathscr{H}$-datum $(\mathbb{P}, \hat{Z}, b)$.

Remark

The Test Function Conjecture [Hai14] specifies a function in $Z(G(\mathbb{Q}_{p^r}); G(\mathbb{Z}_{p^r}))$ which should be plugged into the twisted orbital integrals in the counting points formula. By [Hai-Ric] the test function can be expressed in terms of the geometry of the local model...

Theorem (Local Model Theorem I)

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Fix a global $\nabla\mathscr{H}$-datum $(\mathfrak{G}, \hat{\underline{Z}}, H)$. Assume that $\mathfrak{G}$ is smooth over C. Then there is the following roof

$$
\begin{array}{ccc}
 & \nabla_n^{H,\hat{\underline{Z}}}\mathscr{H}^1(C,\mathfrak{G})_{\underline{R}} & \\
\swarrow^{\bullet} & & \searrow^{loc} \\
\nabla_n^{H,\hat{\underline{Z}}}\mathscr{H}(C,\mathfrak{G})^1_{\underline{R}} & & \prod_i \hat{Z}_{i,R_i},
\end{array}
\qquad (2)
$$

Theorem (Local Model Theorem I)

Fix a global rH -datum $(G; \underline{\hat{Z}}; H)$. Assume that G is smooth over C. Then there is the following roof

$$r_n^{H;\underline{\flat 2}} \, f \, H^1(C;G)_{\underline{R}} \qquad (2)$$

$$r_n^{H;\underline{\flat 2}} H(C;G)^1_{\underline{R}} \qquad \overset{\text{loc}}{} \; \prod_i \flat 2_{i;R_i};$$

Let y be a geometric point of $r_n^{H;\underline{\flat 2}} H^1_{\underline{R}}$. The $\prod_i L^+ P_i$-torsor

$: r_n^{H;\underline{\flat 2}} g H^1_{\underline{R}} \; ! \; r_n^{H;\underline{\flat 2}} H^1_{\underline{R}}$ admits a section s, locally over an etale neighborhood of y, such that the composition $^{\text{loc}}$ s is formally etale.

Theorem (Local Model Theorem II)

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To a local rH -datum $(P;\hat{Z};b)$ one can assign a roof

$$\begin{array}{ccc} & \widetilde{M} & \\ \swarrow & & \searrow^{\text{loc}} \\ M^{\bullet}(P;\hat{Z};b) & & \hat{Z}_{b}; \end{array} \qquad (3)$$

Theorem (Local Model Theorem II)

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$$\tilde{M} \quad (3)$$

$$M^{\bullet}(P; \hat{Z}; b) \longleftarrow \tilde{M} \xrightarrow{\ \ loc\ \ } \hat{Z}_b;$$

that satis es the following properties

1. the morphism ^{loc} is formally smooth and
2. $\tilde{M}$ is an L^+P-torsor under $: \tilde{M} \to M(P; \hat{Z}; b)$. It admits a section s^0 locally for the etale topology on $M(P; \hat{Z}; b)$ such that $^{loc}\ s^0$ is formally etale.

Theorem

Keep the above notation and consider a compact open subgroup $H \subset G(\mathbb{A}_{-})$. Fix $G_0 \in \nabla_n^{H,\hat{Z}} H^1(C, G)(k)$.

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- There is a $I_{\underline{G}_0}(Q)$-invariant morphism

$$^0: \prod_i M^{\hat{Z}_i}_{\underline{L}_i} \quad \mathrm{Isom}\ (! \quad ; ! \ -(\underline{G}_0))=H \ ! \ r^{H;\hat{Z}}_{n} H^1(C;G)- b_{R_{\hat{Z}}} \mathrm{Spf}\, R_{\hat{Z}} \ ; \tag{4}$$

where $I_{\underline{G}_0}(Q)$ acts trivially on the target and diagonally on the source. Furthermore, this morphism factors through

$$: \ I_{\underline{G}_0}(Q) \quad \prod_i M^{\hat{Z}_i}_{\underline{L}_i} \quad \mathrm{Isom}\ (! \quad ; ! \ -(\underline{G}_0))=H \tag{5}$$

$$\#$$

$$r^{H;\hat{Z}}_{n} H^1(C;G)- b_{R_{\hat{Z}}} \mathrm{Spf}\, R_{\hat{Z}}$$

of ind-DM-stacks over $\mathrm{Spf}\, R_{\hat{Z}}$, which is a monomorphism

- Let $\{T_j\}$ be a set of representatives of $\underline{G}_0(\mathbb{Q})$-orbits of the irreducible components of the scheme $\prod_i X_{Z_i}(\underline{L}_i) \times \mathrm{Isom}(\omega, \omega_{\underline{G}_0})/H$ which is locally of finite type over k.
  Then the image $\Theta'(T_j)$ of T_j under Θ' is a closed substack with the reduced structure and each $\Theta'(T_j)$ intersects only finitely many others. Let $\mathcal{Z}$ be the union of the $\Theta'(T_j)$. Its underlying set is the isogeny class of $\underline{G}_0$, that is the set of all $(\underline{G}, \eta H)$ for which $\underline{G}$ is isogenous to $\underline{G}_0$. Let $\nabla_n^{H,\hat{\underline{Z}}}\mathscr{H}^1(C,\mathfrak{G})_{/\mathcal{Z}}^{\wedge}$ be the formal completion of $\nabla_n^{H,\hat{\underline{Z}}}\mathscr{H}^1(C,\mathfrak{G}) \hat{\times}_{R_{\hat{\underline{Z}}}} \mathrm{Spf}\, R_{\hat{\underline{Z}}}$ along $\mathcal{Z}$. Then Θ induces an isomorphism of locally noetherian, adic formal algebraic Deligne-Mumford stacks locally formally of finite type over $\mathrm{Spf}\, R_{\hat{\underline{Z}}}$

(Langlands-Rapoport Conjecture For G-shtukas)

De nition

De ne the functor L which sends a H data $(G; (\hat{Z}_i)_i; H)$ to the disjoint union $\bigsqcup_{\underline{G}} S(\underline{G})$ of the quotient stack

$$S(\underline{G}) := I(Q)(\underline{G}) \quad \prod_i X_{Z_i}(!_i(\underline{G})) \quad X^{-} = H; \tag{6}$$

where $\underline{G}$ runs over the quasi-isog classes of G-shtukas and $X^{-} = \mathrm{Isom}\ (! \ ; !^{-}(\underline{G}))$.

When $\underline{G}$ runs over the quasi-isog classes of special G-shtukas, we denote the resulting functor by L_{spe} (resp. L_{adm}).

We de ne the functor L (resp. L_{spe}, resp. L_{adm}) as the functor which sends $(G; (\hat{Z}_i)_i)$ to

$\lim_H L(G; (\hat{Z}_i)_i; H)$ (resp. $\lim_H L_{spe}(G; (\hat{Z}_i)_i; H)$, resp. $\lim_H L_{adm}(G; (\hat{Z}_i)_i; H)$).

Theorem

There exist a canonical $G(A_{-})$ $\quad Z(Q)$-equivariant isomorphism of functors

$$L\ (\quad) ! r\!\!\sim \quad {}_{n}^{;}\ H^{1}(C;\quad)_{-}(\overline{F}):$$

Moreover via this isomorphism, the operation on the left hand side of the above isomorphism corresponds to the Frobenius endomorphism on the right hand side. Furthermore:

-One may replace $L\ (\quad)$ by $L_{adm}(\quad)$.

-when G is quasi-split, one may further replace $L\ (\quad)$ by $L_{spe}(\quad)$.

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a) There is a group theoretic criterion for admissibility by Xuhua He [HE].

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b) Also the set of isogeny classes and the groups L_0 have a group theoretic description in terms of the Tannakian fundamental group of the category of function eld motives.

c) For $G = GL_n$ the q-isog classes can be described using Honda-Tate theory.

Let $(G; (\hat{Z}_i)_i; H)$ be a rH $-$-data and let $Z_i \quad F \quad \coprod P_i \quad F_i \quad R_{\hat{Z}_i}$ denote the special fiber of $\hat{Z}_i$.

The above observation enables us to describe $]_n^{H;\hat{Z}} - H(F)$ as a sum of terms of the form

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$$\mathrm{vol}(I(Q)\backslash G(A))\, O(1_H) \prod_i TO_i(1_{Z_i})$$

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Here

$$O(1_H) := \int_{G(A)_o \backslash G(A)} 1_H(g^{-1} \ g) d\overline{g}$$

and

$$TO_i(1_{Z_i}) := \int_{I(Q_i) \backslash G(Q_i)} 1_{Z_i}(h_i^{-1} \ {}_i \ (h_i)) d\overline{h}_i ;$$

-$d\overline{x}$ denote the Haar measure on $G(A)$ which gives measure 1 to H.

-1_H denote the characteristic function of H and

-1_{Z_i} denote the characteristic function corresponding for $Z_i \quad F \ `P_i \quad F_i \quad R_{\hat{Z}_i}$.

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Thank you !